INTRODUCTION TO THE RENORMALIZATION GROUP

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1 Ising Model

We discuss first a concrete example of spin systems: the Ising model, and later generalize.

Lattice : $\mathbf{Z}^d$, $x = (x_1, \cdots, x_d) \in \mathbf{Z}^d$, $x_i \in \mathbf{Z}$.

Spins : To each $x \in \mathbf{Z}^d$, associate a *spin variable* $\sigma_x \in \{-1, 1\}$.

The Configuration space of the infinite system is the set of all spin configurations $\Omega = \{-1, 1\}^{\mathbf{Z}^d}$ i.e. $\sigma \in \Omega$ is a map $\sigma : \mathbf{Z}^d \to \{-1, 1\}$ i.e. $x \to \sigma_x$.

Let $\Lambda \subset \mathbf{Z}^d$. Denote $\Omega_\Lambda = \{-1, 1\}^\Lambda$ = configurations in Λ. Let $|\Lambda|$ be the number of elements in Λ.

Example $\Lambda = \{x \in \mathbf{Z}^d \| x_i| \leq \frac{L}{2} = i = 1, \cdots, d\}$ "cube of side L".

Definition Let $|\Lambda| < \infty$. The *Ising Hamiltonian* in Λ is $\mathcal{H}_\Lambda : \Omega_\Lambda \to \mathbf{R}$ given by

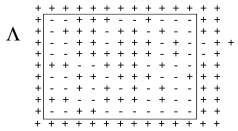
$$\mathcal{H}_\Lambda(\sigma) = -J \sum_{\substack{< xy > \\ x \in \Lambda, y \in \Lambda}} \sigma_x \sigma_y - h \sum_{x \in \Lambda} \sigma_x \tag{1.1}$$

where $J > 0, h \in \mathbf{R}$ and $< xy >$ means we sum over all *nearest neighbour* pairs $x, y \in \Lambda$, $|x - y| = 1$ where we use the norm $|x| = \left(\sum_{i=1}^d x_i^2\right)^{1/2}$.

Motivation : $\mathcal{H}$ takes its smallest value when all σ_x = sgn h (= sign of h i.e. $\frac{h}{|h|}$). I.e. minimum energy when all spins (magnetic moments) are parallel to the field. Ferromagnetism.

Boundary conditions (1.1) is the so called *free boundary condition* Hamiltonian. Only σ_x with $x \in \Lambda$ enter. Other boundary conditions $(b.c)$:

a) $+$ *b.c.* : Take $\mathcal{H}$ as above, but demand that x or y is in Λ. For $\sigma_x, x \notin \Lambda$ let $\sigma_x = 1$.



b) $-$ *b.c* is similair

c) Periodic *b.c.* Replace $\mathbf{Z}^d$ by $(\mathbf{Z}_L)^d$ where $\mathbf{Z}_L = \{0, 1, \cdots, L-1\}$ and nearest neighbours are as above plus the pair $(0, L-1)$[1]. Can also consider $\mathbf{Z}_{L_1} \times \mathbf{Z}_{L_2} \times \cdots \times \mathbf{Z}_{L_d}$.

Denote by $\mathcal{H}_\Lambda^{b.c.}(h, \sigma)$ the Ising Hamiltonian in Λ with *b.c.* = free, $\pm$ or periodic. We also set $J = 1$ since it will play no role below.

Definition The *Ising measure* on Ω_Λ is the measure (note : Ω_Λ is a finite set, $|\Omega_\Lambda| = 2^{|\Lambda}$)

$$\mu_{\beta,h,\Lambda}^{b.c.}(\sigma) = \frac{e^{-\beta \mathcal{H}_\Lambda^{b.c.}}(h, \sigma)}{Z^{b.c.}(\beta, h, \Lambda)}$$

where $Z^{b.c.}$ is the *partition function*:

$$Z^{b.c.}(\beta, h, \Lambda) = \sum_{\sigma \in \Omega_\Lambda} e^{-\beta \mathcal{H}_\Lambda^{b.c.}(h,\sigma)}$$

so μ is a *probability measure*, $\mu(\Omega_\Lambda) = 1$.

Remark In physics, $\beta = \frac{1}{kT}$ where k=Boltzman's constant and T = temperature : β is "inverse temperature". Note that for $\beta \to \infty$ $(T \to 0)$ the minima of $\mathcal{H}$ dominate : in low temperature, magnetism.

In the real world $|\Lambda| \sim 10^{23}$, so we inquire

1° What happens as $\Lambda \to \mathbf{Z}^d$? Is there a limit measure $\mu = \lim_{\Lambda \to \mathbf{Z}^d} \mu_\Lambda$ on Ω? This is called the *thermodynamic limit.*

2° Does μ depend on *b.c.*? How does it depend on β, h?

Study μ_Λ via its *correlation functions* :

Definition Let $A \subset \mathbf{Z}^d, |A| < \infty$. Denote

$$\sigma_A = \prod_{x \in A} \sigma_x.$$

The *correlation function* of spins in A is (let $A \subset \Lambda$)

$$\begin{aligned} \langle \sigma_A \rangle_{\beta,h,\Lambda}^{b.c.} &\equiv \int \sigma_A d\mu_{\beta,h,\Lambda}^{b.c.} \\ &= \frac{1}{Z} \sum_{\sigma \in \Omega_\Lambda} e^{-\beta \mathcal{H}^{b.c.}(h,\sigma)} \sigma_A. \end{aligned}$$

[1] i.e. The cyclic group of order L.

We often drop β, h in notation : $\langle\sigma_A\rangle_\Lambda^{b.c.}$.

Examples $A = \{x\}$, $\langle\sigma_x\rangle_\Lambda^{b.c.}$ is *magnetization* at x. $A = \{x, y\}$, $\langle\sigma_x\sigma_y\rangle_\Lambda^{b.c.}$ is the *pair correlation* or *2-point function*.

2 Reconstruction of the Measure from the Correlation Functions

1. Let $f : \Omega_\Lambda \to \mathbf{R}$. Then $\langle f\rangle_\Lambda = \int f d\mu_\Lambda$ (we drop *b.c.* from notation) is defined, and defines a linear functional ℓ_Λ on the vector space of functions from $\Omega_\Lambda \to \mathbf{R}$. I.e. let $\mathcal{F}_\Lambda = \{f | f : \Omega_\Lambda \to \mathbf{R}\}$.

Then $\ell_\Lambda : \mathcal{F}_\Lambda \to \mathbf{R}$, given by

$$\ell_\Lambda(f) = \langle f\rangle_\Lambda,$$

is a linear map. It satisfies

a) $\ell_\Lambda(1) = 1$

b) If $f \geq 0$ then $\ell_\Lambda(f) \geq 0$

c) $|\ell_\Lambda(f)| \leq \|f\| = \sup_\sigma |f(\sigma)| (= \max |f(\sigma)|)$.

Generally, let M be compact metric space, $C(M) = \{f : M \to \mathbf{R} | f \text{ continuous }\}$. $C(M)$ is a Banach space (= vector space with norm which is complete i.e. Cauchy sequences have limits),

$$\|f\| = \sup_{x\in M} |f(x)|.$$

A linear map $\ell : C(M) \to \mathbf{R}$ is called a *state* on $C(M)$ if a)-c) above hold.

Remark $\mathcal{F}_\Lambda = C(\Omega_\Lambda)$ above since Ω_Λ is a finite set, hence compact, and all functions are continuous.

Theorem. (Rietz Representation Theorem). *Given a state ℓ on $C(M)$ there is a Borel probability measure μ on M such that*

$$\ell(f) = \int f d\mu$$

Proof See Rudin, Real and Complex Analysis

Remark Borel measure : measure on the σ-algebra of Borel sets of M = smallest σ-algebra containing open sets. Clearly $f \to \int f d\mu$ satisfies a)-c). The converse is the hard part !

Exercise Prove this directly for $M = \Omega_\Lambda$!

We can now pose the $\Lambda \to \mathbf{Z}^d$ question as follows:

1. Ω is a compact metric space: $\{-1, 1\}$ is compact, so $\Omega = \{-1, 1\}^{\mathbf{Z}^d}$ is also compact

(Tychonov theorem). A metric compatible with this topology is e.g.

$$d(\sigma,\sigma') = \sum_{x\in\mathbf{Z}^d} 2^{-|x|}|\sigma_x - \sigma'_x|$$

Homework : Check all this !

2. Let $C_0(\Omega) = \{f : \Omega \to \mathbf{R} | f$ depends on finitely many σ_x only $\}$. Thus if $f \in C_0$ then $f(\sigma)$ depends only on $\sigma_x, x \in \Lambda$ some Λ with $|\Lambda| < \infty$. $C_0(\Omega)$ is dense in $C(\Omega)$. Moreover, every $f \in C_0(\Omega)$ is a linear combination of σ_A with $|A| < \infty$ (Example, let f depend on σ_x only. Then $f(\sigma_x) = \frac{1}{2}(f(-1)+f(1)) + \frac{1}{2}\sigma_x(f(1)-f(-1))$. General case similair).

Definition We say that the states $\langle - \rangle_\Lambda$ converge to a state $\langle - \rangle$ on $C(\Omega)$ if for all A, $|A| < \infty$,

$$\langle \sigma_A \rangle_\Lambda \to \langle \sigma_A \rangle \text{ as } \Lambda \to \mathbf{Z}^d$$

Remark 1 We need to specify how $\Lambda \to \mathbf{Z}^d$. We take $\Lambda = L$-cube and $L \to \infty$. This can be generalized.

Remark 2 If the numbers $\langle \sigma_A \rangle_\Lambda$ converge $\forall A$ then we may define a map ℓ on $C_0(\Omega)$:

$$\ell(f) = \lim_{\Lambda\to\mathbf{Z}^d} \langle f \rangle_\Lambda$$

ℓ clearly satisfies:

a) $\ell(1) = 1$.

b) $\ell(f) \geq 0$ if $f \geq 0$.

c) $|\ell(f)| \leq \|f\|$,

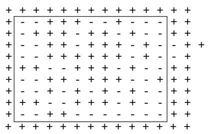
i.e. ℓ is a state on $C_0(\Omega)$. Since $C_0(\Omega)$ is dense in $C(\Omega)$ and ℓ satisfies c), ℓ extends uniquely to $C(\Omega)$ satisfying a)-c); i.e. limits are given by a state on $C(\Omega)$. By Rietz, there is a measure μ on $C(\Omega)$ such that

$$\lim_{\Lambda\to\mathbf{Z}^d} \int \sigma_A d\mu_\Lambda = \int \sigma_A d\mu.$$

We call $\langle - \rangle$ (or μ) *infinite volume state* and the question of thermodynamic limit is : Find all infinite-volume states by taking different *b.c.*. We'll see that for all *b.c.* the $\Lambda \to \mathbf{Z}^d$ limit exists (at least, through subsequences), but there can be many different $\mu's$.

3 $d = 1$ Ising Model : Transfer Matrix

Λ



Let first $h = 0$ and consider $\mathcal{H}$ with, say, $+b.c.$:

$$\beta\mathcal{H} = -\beta \sum_{i=-L}^{L+1} \sigma_{i-1}\sigma_i$$

where $\sigma_{-L-1} = \sigma_{L+1} = 1$.

<u>Partition function</u>:

$$Z_L^+ = \sum_{\sigma_{-L},\cdots,\sigma_L=\pm 1} \quad \prod_{j=-L}^{L+1} e^{\beta\sigma_{i-1}\sigma_i}.$$

Let $T = (T_{\sigma\sigma'})_{\sigma,\sigma'=\pm 1}$ be the 2×2 matrix:

$$(T_{\sigma\sigma'}) = (e^{\beta\sigma\sigma'}) = \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix}.$$

Then,

$$\begin{aligned} Z_L^+ &= \sum_{\sigma_{-L},\sigma_L=\pm 1} e^{\beta\sigma_{-L-i}\sigma_{-L}} (T^{2L})_{\sigma_{-L}\sigma_L} e^{\beta\sigma_L\sigma_{L+1}} \\ &= (f^+, T^{2L} f^+) \quad \text{where} \quad (f,g) \equiv \sum_{\sigma} f_\sigma g_\sigma \end{aligned}$$

and $f^+ = \begin{pmatrix} e^{\beta} \\ e^{-\beta} \end{pmatrix}$.

T is symmetric, has eigenvalues : $(e^{\beta} - \lambda)^2 - e^{-2\beta} = 0 \Rightarrow \lambda = \left\{ \begin{array}{lcl} 2\cosh\beta & = & \lambda_1 \\ 2\sinh\beta & = & \lambda_2 \end{array} \right.$, and orthonormal eigenvectors $e_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, e_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Let P_i be the orthogonal projections on these ($i = 1, 2$). So $P_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, P_2 = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and $T = \lambda_1 P_1 + \lambda_2 P_2$ and $P_1 P_2 = 0, P_i^2 = P_i \Rightarrow$

$$T^{2L} = (2\cosh\beta)^{2L}(P_1 + (\tanh\,\beta)^{2L} P_2)$$

and

$$Z_L^+ = (2\cosh\beta)^{2L} \left[(f^+, e_1)^2 + (\tanh\,\beta)^{2L} (f^+, e_2)^2 \right].$$

Note that $(f^+, e_1)^2 = \frac{1}{2}(e^{\beta} + e^{-\beta})^2 = 2(\cosh\beta)^2 > 0\ \forall\beta$, so

$$Z_L^+ = \exp 2L \left[\log(2\cosh\beta) + \mathcal{O}\left(\frac{1}{L}\right) + \mathcal{O}(e^{-\alpha L}) \right]$$

where $\alpha = \frac{1}{2}\log(\tanh\,\beta) > 0, \quad \forall\beta$.

Definition Free energy in volume Λ is

$$F_\Lambda^{b.c.} = \frac{1}{|\Lambda|} \log Z_\Lambda^{b.c.}.$$

We got in $1d$ Ising:

$$F_{2L}^{+} = \log(2\cosh\beta) + \mathcal{O}(\frac{1}{L}) + \mathcal{O}(e^{-\alpha L})$$

has the $L \to \infty$ limit :

$$F^{+} = \log(2\cosh\beta).$$

Note that $\log Z_{2L}$ is *extensive* i.e., once divided by $|\Lambda|$, has a (thermodynamic) limit.

Dependence on $b.c.$ For $-b.c.$ the only change is $f^{+} \to f^{-} = \begin{pmatrix} e^{-\beta} \\ e^{\beta} \end{pmatrix}$. We see that $F^{+} = F^{-}$.

Homework Show that $F^{free} = F^{perodic} = F^{+} = F^{-}$.

Magnetization Consider

$$\begin{aligned}
\langle \sigma_x \rangle_{2L}^{+} &= \frac{1}{Z_{2L}^{+}} \sum_{\sigma} e^{-\beta \mathcal{H}_{2L}^{+}(\sigma)} \sigma_x = \\
&= \frac{\sum_{\sigma_{-L},\sigma,\sigma_L} f_{\sigma_{-L}}^{+} (T^{L+x})_{\sigma_{-L}\sigma} \sigma (T^{L-x})_{\sigma\sigma_L} f_{\sigma_L}^{+}}{Z_{2L}^{+}} \\
&= \frac{\sum f_{\sigma_{-L}}^{+} (P_1)_{\sigma_{-L}\sigma} \sigma (P_1)_{\sigma\sigma_L} f_{\sigma_L}^{+} + \mathcal{O}(e^{-\alpha(L+x)}) + \mathcal{O}(e^{-\alpha(L-x)})}{(f^{+}, P_1 f^{+}) + \mathcal{O}(e^{-\alpha L})}.
\end{aligned}$$

But $(P_1 f^{+})_{\sigma} =$ constant in $\sigma \left(= \frac{1}{2}(f_{-}^{+} + f_{+}^{+})\right)$ so the first term in the numerator is proportional to $\sum_{\sigma} \sigma = 0$. So

$$|\langle \sigma_x \rangle_{2L}^{+}| \leq C(e^{-\alpha(L+x)} + e^{-\alpha(L-x)}) \underset{L\to\infty}{\longrightarrow} 0.$$

I.e. magnetization vanishes. Also for $\langle \sigma_x \rangle^{-}$.

Correlation functions Let $x < y$:

$$\langle \sigma_x \sigma_y \rangle_{2L}^{+} = \frac{1}{Z_{2L}^{+}} \sum \; f_{\sigma_{-L}}^{+} (T^{L+x})_{\sigma_{-L}\sigma_x} \sigma_x (T^{y-x})_{\sigma_x\sigma_y} \sigma_y (T^{L-y})_{\sigma_y\sigma_L} f_{\sigma_L}^{+}$$

$$= \frac{1}{(f^{+}, P_1 f^{+}) + \mathcal{O}(e^{-\alpha L})} \sum f^{+} (P_1 + e^{-\alpha(L+x)} P_2) \sigma_x (P_1 + e^{-\alpha(y-x)} P_2) \sigma_y (P_1 + e^{-\alpha(L-y)} P_2) f^{+}.$$

Note that $P_1 \sigma P_1 = 0$ since $(P_1)_{\sigma\sigma'} = \frac{1}{2}, \sum_{\sigma} \sigma = 0$.
Thus

$$\langle \sigma_x \sigma_y \rangle_{2L}^{+} = \frac{(f^{+}, P_1 \sigma P_2 \sigma P_1 f^{+}) e^{-\alpha(y-x)} + \mathcal{O}(e^{-\alpha L})}{(f^{+}, P_1 f^{+}) + \mathcal{O}(e^{-\alpha L})}$$

$\underset{L\to\infty}{\longrightarrow} e^{-\alpha(y-x)}$ since $\sigma P_2 \sigma = P_1$ and $P_1^3 = P_1$. We got:

$$\langle \sigma_x \sigma_y \rangle^{+} = e^{-\alpha|y-x|} \qquad \alpha = \log\tanh\beta > 0 \quad \forall \beta.$$

Another way to say this :

$$\langle \sigma_x \sigma_y \rangle^{+} - \langle \sigma_x \rangle^{+} \langle \sigma_y \rangle^{+} = e^{-\alpha|x-y]} \underset{|x-y|\to\infty}{\longrightarrow} 0$$

correlations decay exponentially. (Here $\langle\sigma_x\rangle^+ = 0$) $\frac{1}{\alpha}$ is called the *correlation length* ξ. Note that $\xi \to \infty$ as $\beta \to \infty$. At low temperatures, the $1d$ model gets more correlated. It is easy to show now (exercise) : let $x_1 \langle x_2 \langle \cdots \langle x_{2n}$. Then

$$\langle \prod_{i=1}^{2n} \sigma_{x_i} \rangle_{2L}^{+} \quad \underset{L\to\infty}{\longrightarrow} \; e^{-\alpha \sum_{i=1}^{n}(x_{2i}-x_{2i-1})}$$

i.e.

$$\langle \prod_{i=1}^{2n} \sigma_{x_i} \rangle^{+} = \prod_{i=1}^{n} \langle \sigma_{x_{2i-1}} \sigma_{x_{2i}} \rangle^{+}. \qquad (*)$$

We got

Theorem 4.1 For the $1d$ Ising model the thermodynamic limit exists and is independent on *b.c.* (for all $\beta \in [0,\infty)$. Correlations are given in the ∞-volume state by (*) with

$$\langle \sigma_x \sigma_y \rangle = (\tanh\, \beta)^{|x-y|}$$

Homework Calculate $Z_{2L}^{b.c.}(\beta, h)$ and $F(\beta,h) = \lim_{L\to\infty} \log Z_{2L}^{b.c.}(\beta,h)$. Show : independent of *b.c.* and *analytic* in β, h.

Remark T is called the *transfer matrix*. What was important above was that $(f^{b.c.}, e_1) > 0$. This is a general fact, following from Perron-Frobenius theorem. We'll discuss this later.

Remark: Above means : $1d$ Ising has no phase transition i.e. there is unique ∞ volume state. We'll discuss this later.

4 $d \geq 2$. High Temperature Uniqueness.

$d = 2$ Ising : $\langle \sigma_A \rangle$ can be solved in closed form (Onsager '44). This marked a new era of statistical mechanics : solution showed existence of phase transition (see below).
Exists other exactly solvable models in $d = 2$ (see Baxter's book). However, this is very exceptional and, in $d > 2$, there are practically none.

We'll develop methods to study $\langle \sigma_A \rangle$ for β small (high temperature) and β large (low temperature). These work for all d and much more general models.

Recall : $\Omega = \{-1,1\}^{\mathbf{Z}^d}$. Let $\bar{\sigma} \in \Omega, \Lambda \subset \mathbf{Z}^d, |\Lambda| < \infty$.

Definition Ising Hamiltonian with $\bar{\sigma}$ as boundary condition :

$$\mathcal{H}_\Lambda^{\bar{\sigma}}(\sigma) = - \sum_{\substack{|x-y|=1 \\ x \in \Lambda \text{ or } y \in \Lambda}}^{\bar{\sigma}} \sigma_x \sigma_y - h \sum_{x\in\Lambda} \sigma_x$$

where in $\sigma_x\sigma_y$ we set $\sigma_x = \bar{\sigma}_x$ if $x \notin \Lambda$, y similair.

$\boxed{\Lambda\ \sigma}$ $\qquad\qquad\qquad$ $\bar{\sigma}\ \Lambda^c$

Before we had

$+ : \bar{\sigma}_x = +1,\ \forall x$

$- : \bar{\sigma}_x = -1,\ \forall x.$

Note : $\mathcal{H}^{\bar{\sigma}}$ depends on $\bar{\sigma}_x, x \in \partial\Lambda = \{y \in \Lambda^c | \mathrm{dist}(y, \Lambda) = 1\}$.

Thus

$$\sum{}^{\bar{\sigma}} \sigma_x\sigma_y = \sum_{x,y\in\Lambda} \sigma_x\sigma_y + \sum_{\substack{x\,\in\,\Lambda \\ y\,\in\,\Lambda^c}} \sigma_x\bar{\sigma}_y.$$

Note that to say $\bar{\sigma} = +1$ we have on the boundary of Λ a positive magnetic field that tends to force σ_x to be $+1$. The question is: can this result in a positive $\langle\sigma_x\rangle$ as $\Lambda \to \mathbf{Z}^d$?

Theorem 4.2.
A) $\exists\beta_0 > 0$ such that if $\beta < \beta_0$ then $\langle\sigma_A\rangle^{\bar{\sigma}}_{\beta,h,\Lambda}$ has a limit as $\Lambda \to \mathbf{Z}^d$ (via cubes say), independent on $\bar{\sigma}$.
B) Let $h = 0$, $d \geq 2$. $\exists\beta_1 < \infty$: if $\beta > \beta_1$ then there exists at least two different infinite-volume (pure) states $\langle - \rangle^+ \neq \langle - \rangle^-$.

Remark

A) means uniqueness of the Gibbs state, see below.

B) means phase transition. See next section. In $d = 2$ only 2 (pure) states, in $d \geq 3$ more.

Finally, one expects (and can be proved, see Simon's book) that $\beta_0 = \beta_1$.

We prove A) in this section.

4.1 High Temperature Expansion

When $\beta = 0$, $\mu^{b.c.}_\Lambda$ is a product measure

$$\mu^{b.c.}_\Lambda(\sigma) = \prod_{x\in\Lambda} \frac{e^{h\sigma_x}}{e^h + e^{-h}} \equiv \prod_{x\in\Lambda} \nu_h(\sigma_x).$$

Thus

$$\langle\sigma_A\rangle^{b.c.}_\Lambda = \left(\sum_{\sigma=\pm1} \sigma\nu_h(\sigma)\right)^{|A|} = (\tanh\ h)^{|A|}$$

factorizes and is Λ-independent. We wish to show that for small β our measure approximately factorizes. Let $h = 0$ for simplicity. Write (since $\sigma_x\sigma_y \in \{-1, 1\}$)

$$e^{\beta\sigma_x\sigma_y} = \cosh\beta + \sigma_x\sigma_y \sinh\beta = \cosh\beta(1 + \sigma_x\sigma_y \tanh\,\beta).$$

Note that for β small $\tanh\beta$ is small. Consider e.g.

$$\langle\sigma_x\sigma_y\rangle_\Lambda = \sum \sigma_x\sigma_y e^{-\beta\mathcal{H}} / \sum e^{-\beta\mathcal{H}}$$

<u>Notation</u> *A bond* is a nearest neighbour pair $\{x, y\}, x, y \in \mathbf{Z}^d, |x - y| = 1$. We may picture a bond by a line between the points $x - - - - y$. It is a special subset of $\mathbf{Z}^d$.

Let us first discuss the *free b.c.* case. With this notation

$$\mathcal{H}_\Lambda^{free}(\sigma) = -\sum_{b\subset\Lambda} \sigma_b$$

The sum is over all bonds in Λ. Denote : $\mathcal{B}_\Lambda = \{$ bonds in $\Lambda\}$. Thus ($|\mathcal{B}_\Lambda| = \#$ bonds in Λ)

$$e^{-\mathcal{H}_\Lambda^{free}(\sigma)} = (\cosh\beta)^{|\mathcal{B}_\Lambda|} \prod_{b\subset\Lambda} (1 + \sigma_b \tanh\beta).$$

Insert this in $\langle\sigma_x\sigma_y\rangle_\Lambda$, cancel $(\cosh\beta)^{|\mathcal{B}_\Lambda|}$ in numerator and denominator and expand the product over b:

$$\langle\sigma_x\sigma_y\rangle_\Lambda^{free} = \frac{\sum_{\sigma\in\Omega_\Lambda} \sum_B \sigma_x\sigma_y \prod_{b\in B} \sigma_b(\tanh\beta)^{|B|}}{\sum_{\sigma\in\Omega_\Lambda} \sum_B \prod_{b\in B} \sigma_b(\tanh\beta)^{|B|}}$$

where B run through all families of bonds in Λ, i.e. $B \subset \mathcal{B}_\Lambda$.

<u>Notation</u> Given B, a family of bonds, let ∂B denote the set of sites $x \in \Lambda$ that occur an *odd* number of times in the bonds in B.

<u>Ex</u>

PICTURE

Since $\sum_{\sigma=\pm} \sigma^n = 0$ if n is odd, we get

$$\langle\sigma_x\sigma_y\rangle_\Lambda^{free} = \frac{\sum_{B:\partial B=\{x,y\}} (\tanh\beta)^{|B|}}{\sum_{B:\partial B=\emptyset} (\tanh\beta)^{|B|}}$$

This is the starting point of the high temperature expansion. We use it to prove the existence of a finite correlation length.

Theorem 4.3. *There exists $\beta_0 > 0$ such that for $\beta < \beta_0$*

$$|\langle\sigma_x\sigma_y\rangle_\Lambda^{free}| \leq Ce^{-|x-y|/\xi}$$

uniformly in Λ (i.e. C, $\xi < \infty$, Λ-independent).

Proof Each B with $\partial B = \{x,y\}$ contains a subset of bonds $P = \{b_i\}_{i=1}^n$ $b_i = \{x_i, x_{i+1}\}, x_1 = x, x_{n+1} = y$ (if not, $\partial B \neq \{x,y\}$, show this!).

PICTURE

Thus every B, in the numerator can be decomposed as $B = P \bigcup B'$ where P is some set as above and $\partial B' = \emptyset$. Given B, P is not unique, but let us choose it arbitarity, and call the choice $P(B)$.
We may then rewrite the numerator as

$$\sum_P (\tanh\beta)^{|P|} \quad \sum_{B'}^{P} (\tanh\beta)^{|B'|}$$

where $\sum^P$ is over B' such that $\partial B' = \emptyset$ and $P(P \bigcup B') = P$. Since the $\sum\limits_{B'}^{P}$ runs over a subset of the sum $\sum\limits_B$ in the denominator, we have

$$\sum_{B'}^{P} (\tanh\beta)^{|B'|} / \sum_B (\tanh\beta)^{|B|} \leq 1$$

and so

$$0 \leq \langle \sigma_x \sigma_y \rangle_\Lambda \leq \sum_P (\tanh\beta)^{|P|} \tag{4.1}$$

where P runs through nearest neighbour paths on Λ from x to y, i.e. $P = \{(x_i, x_{i+1})\}_{i=1}^n$ as above, n arbitary. Given x_i there are at most $2d$ nearest neighbour x_{i+1} and n is at least $|x-y|$. Thus

$$(4.1) \leq \sum_{n \geq |x-y|} (2d \tanh\beta)^n \leq C e^{-|x-y|/\xi}$$

if $2d \tanh\beta < 1$

<u>Remark</u> We could have chosen P *self-avoiding* and so could have had only $2d-1$ choices: $x_{i+1} \neq x_{i-1}$. Thus $(2d-1)\tanh\beta < 1$ suffices.

<u>Remark</u>

A) In $d = 2$, $\tanh\beta_c = \sqrt{2} - 1 = 0,4142$
We got $\tanh\beta_c < \frac{1}{3}$ or $\beta_c > 0.333$ (not too bad)

B) As $d \to \infty$, our estimate $\tanh\beta_c \sim 1/2d-1$ becomes exact.
We now slightly modify the previous argument to prove part A) of Theorem 4.2.

Proof of (A) We prove uniqueness (independence on b.c) of the limit. Existence is analogous. Consider $\bar\sigma, \bar\sigma' \in \Omega$ and

$$\begin{aligned}\langle\sigma_A\rangle_\Lambda^{\bar\sigma} - \langle\sigma_A\rangle_\Lambda^{\bar\sigma'} &= \frac{\sum_\sigma e^{-\beta\mathcal{H}_\Lambda^{\bar\sigma}(\sigma)}\sigma_A}{\sum_\sigma e^{-\beta\mathcal{H}_\Lambda^{\bar\sigma}(\sigma)}} - \text{(same with primes)} \\ &= \sum_{\sigma,\sigma'} e^{-\beta\mathcal{H}_\Lambda^{\bar\sigma}(\sigma)-\beta\mathcal{H}_\Lambda^{\bar\sigma'}(\sigma')}(\sigma_A-\sigma'_A)/\sum_{\sigma,\sigma'} e^{-\beta\mathcal{H}_\Lambda^{\bar\sigma}(\sigma)-\beta\mathcal{H}_\Lambda^{\bar\sigma'}(\sigma')} \end{aligned} \tag{4.2}$$

We expand slightly differently (to understand why, see Remark after the Proof).

$$e^{\beta(\sigma_x\sigma_y+\sigma'_x\sigma'_y)} = e^{-2\beta}(1+f_{xy})$$

$$f_{xy} = e^{\beta(\sigma_x\sigma_y+\sigma'_x\sigma'_y+2)} - 1$$

Note that

$$0 \le f_{xy} \le 4\beta(1+f_{xy}) \tag{4.3}$$

using $0 \le e^x - 1 \le xe^x$. Expand in powers of f as before, but only in the numerator: let $\bar{\mathcal{B}}_\Lambda$ be the set of bonds intersecting Λ (i.e. $b=\{x,y\}$, either x or y is in Λ), then

$$\langle\sigma_A\rangle_\Lambda^{\bar\sigma} - \langle\sigma_A\rangle_\Lambda^{\bar\sigma'} = \frac{\sum_{B\subset\bar{\mathcal{B}}_\Lambda}\sum_{\sigma\sigma'}(\sigma_A-\sigma'_A)\prod_{b\in B} f_b}{\sum_{\sigma\sigma'}\prod_{b\in\bar{\mathcal{B}}_\Lambda}(1+f_b)} \tag{4.4}$$

Let us say that $b\in B$ is connected in B to A if $\exists\{b_i\}_{i=1}^n$, $b_i \in B$ $i=1,\cdots,n$ such that $b_i=\{x_i,x_{i+1}\}, b_1 = b$ and $x_{n+1}\in A$.

PICTURE

Let $B_0 \subset B$ be the set of b in B connected to A. Suppose that no bond in B_0 intersects Λ^c (i.e. all are in Λ i.e. all f_b involve σ, σ' and not $\bar\sigma,\bar\sigma'$). Then that term in (4.4) vanishes. Indeed, the sum over σ_x, σ'_x with $x\in\cup_{b\in B_0} b \cup A$ factorizes out and vanishes, since $\prod_{b\in B_0} f_b$ is symmetric under the interchange of σ and σ' whereas $\sigma_A - \sigma'_A$ is antisymmetric.
Hence, the nonvanishing terms in (a) have B's such that B includes a *connected path P of bonds joining A to Λ^c*.
Bound the numerator as

$$|(\sigma_A-\sigma'_A)\prod_{b\in B} f_b| \le 2\prod_{b\in P} f_b \prod_{b\in B\backslash P} f_b \le 2(4\beta)^{|P|}\prod_{b\in P}(1+f_b)\prod_{b\in B\backslash P} f_b$$

using (4.3) in the last inequality (Recall that $f_b \ge 0$!!); and so, picking, as in the proof of Theorem 4.2, for each B, a choice $P(B)$ of P,

$$|\sum_B\sum_{\sigma,\sigma'}(\sigma_A-\sigma'_A)\prod_{b\subset B} f_b| \le 2\sum_P (4\beta)^{|P|}\sum_{\sigma\sigma'}\sum_{B'}^{P}\prod_{b\in P}(1+f_b)\prod_{b\in B'} f_b \tag{4.5}$$

where the sum over B' runs over $B'\subset\bar{\mathcal{B}}_\Lambda\backslash P$ with the condition $P(B'\cup B) = P$. Now,

$$\sum_{B'}^{P}\prod_{b\in B'} f_b \le \sum_{B'\subset\bar{\mathcal{B}}_\Lambda\backslash P}\prod f_b = \prod_{b\in\bar{\mathcal{B}}_\Lambda\backslash P}(1+f_b)$$

so

$$(4.5) \le 2\sum_P (4\beta)^{|P|} \sum_{\sigma\sigma'} \prod_{b\in\bar{\mathcal{B}}_\Lambda} (1+f_b)$$

and inserting into (4.2), we get (after cancelling the denominator)

$$\begin{aligned} |\,\langle\sigma_A\rangle_\Lambda^{\bar\sigma} - \langle\sigma_A\rangle_\Lambda^{\bar\sigma'}| &\le 2\sum_P (4\beta)^{|P|} \le 2\sum_{x\in A}\sum_{y\in\partial\Lambda}\sum_{P:x\to y}(4\beta)^{|P|} \\ &\le C\sum_{x\in A}\sum_{y\in\partial\Lambda} e^{-|x-y|/\xi} \le C|A|\,|\partial\Lambda| e^{-\text{dist }(A,\partial\Lambda)} \end{aligned} \tag{4.6}$$

($\partial\Lambda$ is the boundary of Λ). For $\Lambda = L-$Box, $|\partial\Lambda| \propto L^{d-1}$, dist $(A,\partial\Lambda) \propto L$, (4.6) $\le C|A|L^{d-1}e^{-\alpha L} \to 0$ as $L\to\infty$. We are done.

Remark Note that the positivity of f_{xy} was crucial in (4.5). This is why we did not use the previous way. This method generalizes to a very general class of Hamiltonians.

Homework :

1. Estimate $|\langle\sigma_A\rangle_\Lambda^{\bar\sigma} - \langle\sigma_A\rangle_{\Lambda'}^{\bar\sigma}|$ in the same way:

$$\langle\sigma_A\rangle_\Lambda^{\bar\sigma} - \langle\sigma_A\rangle_{\Lambda'}^{\bar\sigma} = \frac{\sum_{\sigma\in\Omega_\Lambda}\sum_{\sigma'\in\Omega_{\Lambda'}} e^{-\beta\mathcal{H}(\sigma)}e^{-\sigma\mathcal{H}(\sigma')}(\sigma_A - \sigma'_A)}{\sum e^{-\beta\mathcal{H}(\sigma)-\beta\mathcal{H}(\sigma')}}$$

Expand $e^{-\beta\mathcal{H}(\sigma)} = e^{-\beta}\prod_b(1+(e^{\beta\sigma_x\sigma_y}-1), e^{-\beta\mathcal{H}(\sigma')}$ similarly and note that B connecting A to Λ^c only contribute (take $\Lambda\subset\Lambda'$). Use this to show that the limit as $\Lambda \nearrow \mathbf{Z}^d$ exists.

2. Similarly, write

$$|\langle\sigma_A\sigma_B\rangle_\Lambda^{\bar\sigma} - \langle\sigma_A\rangle_\Lambda^{\bar\sigma}\langle\sigma_B\rangle_\Lambda^{\bar\sigma}| = \frac{1}{2}\sum_{\sigma\sigma'} e^{-\beta\mathcal{H}(\sigma)-\beta\mathcal{H}(\sigma')}[\sigma_A-\sigma'_A][\sigma_B-\sigma'_B]/\sum_{\sigma\sigma'} e^{-\beta\mathcal{H}(\sigma)-\beta\mathcal{H}(\sigma')},$$

expand $e^{-\beta\mathcal{H}(\sigma)-\beta\mathcal{H}(\sigma')}$ and prove:

$$\langle\sigma_A\sigma_B\rangle_\Lambda^{\bar\sigma} - \langle\sigma_A\rangle_\Lambda^{\bar\sigma}\langle\sigma_B\rangle_\Lambda^{\bar\sigma}| \le Ce^{-\alpha\text{ dist }(A,B)}$$

i.e. all correlations functions decay exponentially.

4.2 Low Temperature Expansions

Consider $\mathcal{H}_\Lambda^+(\sigma) = -\sum_b \sigma_b$. For $\beta\to\infty$, $e^{-\beta\mathcal{H}}$ reaches its maximum when all $\sigma_b = \sigma_x\sigma_y = 1$ i.e. if all $\sigma_x = +1$ or all $\sigma_x = -1$. Consider, say in $d=2$, the configuration $\sigma_x = 1\ \ \forall x\ne x_0$, $\sigma_{x_0} = -1$:

$$\begin{matrix} + & + & + \\ + & - & + \\ + & + & + \end{matrix}$$

This has energy $\beta\mathcal{H} = \beta\mathcal{H}_{\text{minimum}} + 4\beta \quad (2d\beta$ in general dimension).
Consider a connected region R of $-$:

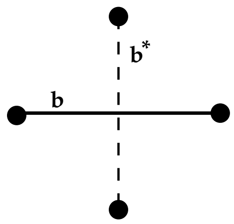
$$\begin{array}{ccccc} + & + & + & + & + \\ + & - & - & - & + \\ + & - & - & - & + \\ + & - & - & - & + \\ + & + & + & + & + \end{array}$$

This has $\mathcal{H} = \mathcal{H}_{\min} + |\partial R|$ where $|\partial R| = \#$ bonds $\{x, y\}$ with $x \in R, y \in R^c$. We say that there is a *contour* around R. Let us formalize this. Given $\sigma \in \Omega_\Lambda$, let

$$C(\sigma) = \{b | b \cap \Lambda \neq \emptyset, \sigma_b = -1\}$$

i.e. all bonds in Λ or joining Λ to Λ^c where $\sigma_x \neq \sigma_y$.

The *dual lattice* of $\mathbf{Z}^d$ is the lattice (we consider both imbedded in $\mathbf{R}^d$) $\mathbf{Z}^d + (\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}) = \{(x_1 \cdots x_d) | \; x_i - \frac{1}{2} \in \mathbf{Z}, \; i = 1, \cdots, d\}$.
Let us view bonds $b = \{x, y\}$ as the closed line segments in $\mathbf{R}^d$ from x to y PICTURE. Similarly PICTURE is a $\{x, y, z, u\}$ defining a 2-cell, PICTURE 3-cell etc. 2-cells are often called *plaquettes*.
Now a bond b in $\mathbf{Z}^d$ defines a unique $(d-1)$-cell b^* "orthogonal"to it in the dual lattice :

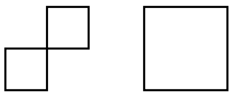


Let us work for simplicity in $d = 2$ and $d = 3$. Let $C^*(\sigma)$ be the union $\cup_{b \in C(\sigma)} b^*$ of dual bonds $(d = 2)$ or plaquettes $(d = 3)$.

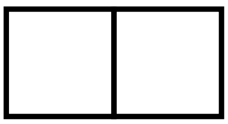
Definition *A contour* γ is a *connected* union of bonds b such that each x in the dual lattice belongs to an even (0,2,4) number of b^*, with $b^* \in \gamma$.

<u>Example</u>

Allowed:



Not allowed:



Lemma $C^*(\sigma)$ has a unique decomposition into a dispoint set of contours $C^*(\sigma) = \cup_\alpha \gamma_\alpha$. Conversely, given such a set, there is a unique σ such that $C^*(\sigma) = \cup \gamma_\alpha$.

Proof

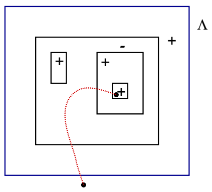
1. $C^*(\sigma)$ has the property that x belongs to an even number of bonds:

PICTURE

We have since $\sigma^2 = 1, \sigma_u\sigma_v \cdot \ \sigma_v\sigma_w\sigma_w\sigma_y \ \sigma_y\sigma_u = 1 = \sigma_{b_1}\sigma_{b_2}\sigma_{b_3}\sigma_{b_4} \Rightarrow$ even number of $\sigma_{bi} = -1 \Rightarrow$ even number of b_i belong to $C(\sigma) \Rightarrow$ claim. Now γ_i are the connected components of $C^*(\sigma)$.

2. Given $\{\gamma_i\}$, let $x_0 \in \Lambda^c$ (so $\sigma_{x_0} = +1$) and let P be any path of bonds from x_0 to x. Let $N(P)$ be the number of $b \in P$ with $b \in C(\sigma)$, i.e. the number of times P crosses contours. Put $\sigma_x = (-1)^{N(P)}$.

<u>Show</u>: If P' is another path, $N(P) - N(P') =$ even.



The Lemma implies:

$$-\beta\mathcal{H}_\Lambda^+(\sigma) = \beta N_\Lambda - 2\beta\sum_\alpha |\gamma_\alpha|$$

($|\gamma_\alpha| = \#$ bonds in γ_α)
and

$$Z_\Lambda^+ = e^{\beta N_\Lambda}\sum_{\{\gamma_\alpha\}}\prod_\alpha e^{-2\beta|\gamma_\alpha|} \tag{4.7}$$

sum over compatible families of contours. Also,

$$\langle\sigma_A\rangle_\Lambda^+ = \frac{\sum_{\{\gamma_\alpha\}}\prod_\alpha e^{-2\beta|\gamma_\alpha|}\prod_{x\in A}(-1)^{N_x(\gamma)}}{\sum_{\{\gamma_\alpha\}}\prod e^{-2\beta|\gamma_\alpha|}} \tag{4.8}$$

$N_x(\gamma) = \#$ of γ_α's surrounding x.

<u>$d = 3$</u> A contour γ is a connected union of plaquettes P which forms a simple closed surface (every edge of every P has 2 plaquettes sharing it). γ, γ' compatible if intersect at most on finite # of edges.
The Lemma generalizes as well as (4.7) and (4.8).

4.3 Magnetization

We prove : for large β,

$$\begin{array}{rcl} \langle\sigma_0\rangle_\Lambda^+ & \geq & \frac{1}{2} \ \forall\Lambda \\ \langle\sigma_0\rangle_\Lambda^- & \leq & -\frac{1}{2} \ \forall\Lambda \end{array}$$

Proof

$$\begin{array}{rcl} \langle\sigma_0\rangle_\Lambda^+ & = & \text{Prob } (\sigma_0 = 1) - \text{ Prob } (\sigma_0 = -1) \\ & = & 1 - 2 \text{ Prob } (\sigma_0 = -1) \end{array}$$

and

Prob $(\sigma_0 = -1)$ = Prob (there is an odd number of contours surrounding 0)

$\leq$ Prob ($\exists$ contour surrounding 0)

$$= \frac{\sum'_{\{\gamma_\alpha\}} \prod_\alpha e^{-\beta|\gamma_\alpha|}}{\sum_{\{\gamma_\alpha\}} \prod_\alpha e^{-\beta|\gamma_\alpha|}}$$

where $\sum'$ has the constraint: there exists a γ_α, say γ_1, such that γ_1 surrounds 0.

$$= \sum_{\substack{\gamma \\ \gamma \text{ surrounds } 0}} e^{-\beta|\gamma|} \sum_{\substack{\{\gamma_\alpha\} \\ \gamma \cup \{\gamma_\alpha\} \\ \text{compatible}}} \prod_\alpha e^{-\beta|\gamma_\alpha|} / \sum_{\{\gamma_\alpha\}} \prod_\alpha e^{-\beta|\gamma_\alpha|}$$

$$\leq \sum_{\gamma \text{ surr. } 0} e^{-\beta|\gamma|} = \sum_{n=2d}^\infty e^{-\beta n} \#\{\gamma : \gamma \text{ surrounds } 0, |\gamma| = n\} \tag{4.9}$$

and estimate (4.9) by :

(4.9) $\leq \ Cn^d$ (# of b^* of distance $\leq n$ from 0).
$(2d)^{2n}$ (starting from b^* make a connected path of $2n$ steps(WHY?), $2d$ choices at each step)
$\leq \ e^{c'n}$ $(n \geq 2d)$

$$\Rightarrow \text{ Prob } (\sigma_0 = -1) \leq \sum_{n=2d}^\infty e^{-(\beta - c')n} < \frac{1}{4} \text{ if } \beta > \beta_0$$

$$\Rightarrow \langle\sigma_0\rangle_\Lambda^+ \geq 1 - 2.\frac{1}{4} = \frac{1}{2}$$

$\langle\sigma_0\rangle_\Lambda^-$ analogous

Remark 1 $\langle\sigma_0\rangle \neq 0$ is called *spontaneous symmetry breaking*: $\mathcal{H}_\Lambda(\sigma)$ is invariant (if $h = 0$) under $\sigma \to -\sigma$ except for the *b.c.* ($\mathcal{H}_\Lambda^{free}$ and $\mathcal{H}_\Lambda^{per}$ are). As $\Lambda \nearrow \mathbf{Z}^d$ the *b.c.* have a finite effect ! Another way to state this : Let $h \neq 0$. Construct $\lim_{\Lambda\nearrow\mathbf{Z}^d}\langle - \rangle_{\Lambda,h}^{\bar\sigma} \equiv \langle - \rangle_h$. It will be independent on $\bar\sigma$ (see below). Then $\lim_{h\downarrow 0}\langle - \rangle_h = \langle - \rangle^+, \lim_{h\uparrow 0}\langle - \rangle^-$. In particular, let $m(h) = \langle\sigma_0\rangle_h$. Then $m(h)$ is *discontinuous* at $h = 0$. This is called a first order phase transition.

Remark 2 The existence of the limit $\Lambda \nearrow \mathbf{Z}^d$ can be proved along the same lines as for β small above.

4.4 High and Low Temperature Expansions

One can go further and get identities and not only inequalities.
Return to high T:

$$\langle \sigma_x \sigma_y \rangle_\Lambda^{free} = \frac{\sum_{\partial B = \{x,y\}} (\tanh \beta)^{|B|}}{\sum_{\partial B = \emptyset} (\tanh \beta)^{|B|}}$$

Numerator : PICTURE
Denominator : PICTURE
Try to cancel the so-called "vacuum graphs". Write $B = \bigcup_\alpha B_\alpha, B_\alpha$ <u>connected</u> sets of bonds. In the numerator, we have one component, say B_1, such that $\partial B_1 = \{x, y\}$. So

$$= \frac{\sum_{B \text{ connected } \partial B = \{x,y\}} \rho(B) \sum_{\{B_\alpha\}}^{B} \prod_\alpha \rho(B_\alpha)}{\sum_{\{B_\alpha\}} \prod_\alpha \rho(B_\alpha)}$$

$\rho(B_\alpha) = (\tanh\ \beta)^{|B_\alpha|}$ where $\sum^B$ is over disjoint B_α's, B_α connected and $B_\alpha \cap B = \emptyset$, in denominator disjoint B_α's connected. The cancellation is performed by the following trick.
Consider the partition function

$$Z = \sum_{n=0}^{\infty} \sum_{\substack{\{B_\alpha\}_{\alpha=1}^n \\ \text{disjoint}}} \prod_{\alpha=1}^{n} \rho(B_\alpha) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{(B_1, B_2, \cdots, B_n) \\ B_i \cap B_j = \emptyset}} \prod_{\alpha=1}^{n} \rho(B_\alpha) \tag{4.10}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{(B_1, \cdots, B_n) \\ (\text{no constraint})}} \prod_{\alpha=1}^{n} \rho(B_\alpha) \prod_{\alpha<\beta} \chi(B_\alpha, B_\beta) \tag{4.11}$$

where $\chi(B_\alpha, B_\beta) = \begin{cases} 0 & B_\alpha \cap B_\beta \neq \emptyset \\ 1 & B_\alpha \cap B_\beta = \emptyset \end{cases}$.

Write $\chi(B_\alpha, B_\beta) = 1 - \eta(B_\alpha, B_\beta)$, $\eta(B_\alpha, B_\beta) = \begin{cases} 0 & B_\alpha \cap B_\beta = \emptyset \\ 1 & B_\alpha \cap B_\beta \neq \emptyset \end{cases}$

$$\prod_{\alpha<\beta} (1 - \eta(B_\alpha, B_\beta)) = \sum_{\mathcal{G}} (-1)^{|\mathcal{G}|} \prod_{(\alpha,\beta)\in\mathcal{G}} \eta(B_\alpha, B_\beta) \tag{4.12}$$

where $\mathcal{G}$ is a collection of pairs (α, β) $\alpha, \beta = 1, \cdots, n, \alpha < \beta$. $\mathcal{G}$ can be identified with a *graph* consisting of *vertices* $\alpha \in v(\mathcal{G}) = \{\alpha | (\alpha, \beta) \in \mathcal{G} \text{ or} (\beta, \alpha) \in \mathcal{G} \text{ for some } \beta\}$ and *lines* $(\alpha, \beta) \in \mathcal{G}$ that connect vertices.
$\mathcal{G}$ is *connected* if any two vertices α, β can be connected by paths in $\mathcal{G}$. Decompose $\mathcal{G}$ into connected components $\mathcal{G} = \bigcup_{i=1}^{k} \mathcal{G}_i$, $\mathcal{G}_i$ connected. Note that this forces the corresponding B_α's to form a conneted network PICTURE. Let $C_i = \bigcup_{\alpha \in v(\mathcal{G}_\alpha)} B_\alpha$.
Reorganize the sum (2) with (3),(4) inserted :

sum over k and over families of k sets $(C_1, \cdots, C_k)$.

sum over $n \geq k$

sum over all families of k subsets of $\{1, \cdots, n\}$ (i.e. $v(\mathcal{G}_i)$ above) sum over B_α's

sum over conneted graphs on those subsets.

So, one gets :

$$Z = \sum_{k=0}^{\infty} \sum_{(C_1,\cdots,C_k)} \sum_{n \geq k} \frac{1}{n!} \sum_{\substack{(n_1,\cdots,n_k) \\ \sum_i n_i = n}} \frac{n!}{n_1! \cdots n_k!} \frac{1}{k!} (*)$$

$$\cdot \prod_{i=1}^{k} \left(\sum_{\substack{(B_1,\cdots,B_{n_i}) \\ \bigcup_{j=1}^{n_i} B_j = C_i}} \sum_{\substack{\mathcal{G} \\ \text{connected graph on} \\ \{1,\cdots,n_i\}}} \prod_{\alpha=1}^{n_i} \rho(B_\alpha) \prod_{(\alpha,\beta)\in\mathcal{G}} (-\eta(B_\alpha, B_\beta) \right)$$

where (*) came from :

$$\begin{pmatrix} n \\ n_1 \end{pmatrix} \begin{pmatrix} n - n_1 \\ n_2 \end{pmatrix} \begin{pmatrix} n - n_1 - n_2 \\ n_3 \end{pmatrix} \cdots \begin{pmatrix} n - \sum_{i=1}^{n-1} n_i \\ n_k \end{pmatrix} \frac{1}{k!} \left(\frac{1}{k!} \text{ because order of the sets does not matter}\right)$$

$\begin{pmatrix} n \\ n_1 \end{pmatrix}$ comes from choosing the n_1 elements of set 1

$$= \frac{n!}{n_1!(n-n_1)!} \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \cdots \frac{1}{k!} = \frac{n!}{\prod n_i!} \frac{1}{k!}$$

So

$$Z = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{(C_1 \cdots C_k)} \sum_{(n_1 \cdots n_k)} \prod_{i=1}^{k} \frac{1}{n_i!} \sum_{\substack{(B_1,\cdots,B_{n_i}) \\ \bigcup_{j=1}^{n_i} B_j = C_i}} \sum_{\substack{\mathcal{G} \\ \text{connected on} \\ \{1,\cdots,n\}}} \prod \rho \prod (-\eta)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_C f(C) \right)^k = \exp \sum f(C)$$

with

$$f(C) = \sum_n \sum_{\substack{(B_1,\cdots,B_n) \\ \bigcup B_j = C}} \frac{1}{n!} \sum_{\substack{\mathcal{G} \\ \text{connected on} \\ \{1,\cdots,n\}}} \prod \rho(B_i) \prod_{(\alpha,\beta)\in\mathcal{G}} (-\eta(B_\alpha, B_\beta)$$

We have achieved our goal. With our assumptions, the exponent is $\mathcal{O}(\Lambda)$. The main estimate is

Lemma Let $|\rho(B)| \leq \epsilon^{|B|}$. There exists ϵ_0 $s.t$ $\epsilon < \epsilon_0 \Rightarrow |f(C)| \leq (2\epsilon)^{|C|}$.
(2 is arbitary here). (Recall, B is always connected here).

Proof Omitted, see Brydges or Seiler .

Remark 3 Note that this is non-trivial. We can not estimate in (4.4) $|-\eta| \leq 1$. Indeed, let C be a single bond. The # of graphs on $\{1,\cdots,n\}$ is the number of subsets of $\{1,\cdots,n\} \times$

$\{1, \cdots, n\}$ i.e. 2^{n^2}, the # of connected graphs is also $\geq e^{an^2}$ $a > 0$. Thus, the nth summand would be $\epsilon^n e^{an^2}$, so does not converge. One needs to account for cancellations. Note that $\sum_{\text{all}\,\mathcal{G}} \prod(-\eta) = \prod \chi(B_\alpha, B_\beta)$ which is ≤ 1. Connectedness makes things harder.
Given the Lemma, write

$$\begin{aligned}
\sum_{B \subset \mathcal{B}_\Lambda} f(B) &= \sum_{b \in \mathcal{B}_\Lambda} \sum_{B : b \in B} \frac{1}{|B|} f(B) \qquad \text{(Why?)} \\
&\equiv \sum_{b \in \mathcal{B}_\Lambda} f_b
\end{aligned}$$

then

$$|f_b| \leq \sum_{n=1}^{\infty} \frac{1}{n} (2\epsilon)^n S_n$$

$S_n = \#$ of connected sets $B, |B| = n$, containing a given bond b.

Proposition $|S_n| \leq (2d)^{2n}$.

Proof We need the Köningsberg bridge Lemma :

Lemma Let G be a finite connected graph and let α_0 be a vertex of G. Then there is a path starting and ending at α_0 which includes each line of G only twice.

Proof Simon, page 464 .

Then all such connected sets are covered by making nearest neighbour walks of length $2n$ i.e. $(2d)^{2n}$ such walks

Thus, for ϵ small $f_b < C < \infty$ and $|\log Z_\Lambda| \leq C|\Lambda|$ so the free energy $\frac{1}{|\Lambda|} \log Z_\Lambda$ is bounded.

Actually, one easily proves that $\lim_{\Lambda \nearrow \mathbf{Z}^d} \frac{1}{|\Lambda|} \log Z_\Lambda = \lim_{\Lambda \nearrow \mathbf{Z}^d} f_b = f$ independently of b.

Also,

$$\begin{aligned}
\langle \sigma_x \sigma_y \rangle_\Lambda^{\text{free}} &= \frac{\sum_{\partial B = \{x,y\}} \rho(B) \exp\left(\sum_{C \cap B = \emptyset} f(C)\right)}{\exp(\sum_C f(C))} \\
&= \sum_{\partial B = \{x,y\}} \rho(B) \exp \sum_{C : C \cap B \neq \emptyset} f(C) \equiv \tilde{\rho}(B) \qquad (4.13)
\end{aligned}$$

and $|\tilde{\rho}(B)| \leq \epsilon^{|B|} e^{C\epsilon|B|} \leq (2\epsilon)^{|B|}$. (4.13) is a version of the high temperature expansion. It can be further processed but we won't bother.

4.5 Ising model-summary

One expects the following *phase diagram* for the Ising model.

PICTURE

I.e. if $h \neq 0$ there is a unique limit of $\langle - \rangle_\Lambda^{\bar{\sigma}}$; if $h = 0$, $\beta \in (\beta_c, \infty)$ 2 states, $\beta \leq \beta_c$ 1 state. The correlation length $\xi(h, \beta) < \infty$ except at $h = 0, \beta = \beta_c$. I.e.

$$| < \sigma_A \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle | \leq C(A, B) e^{-d(A,B)/\xi}.$$

At $h = 0, \beta = \beta_c$ one expects

$$\langle \sigma_x \sigma_y \rangle \sim_{|x-y| \to \infty} \frac{c}{|x-y|^a}$$

At	$d = 2$,	$a = 1/4$	(exact)
	$d = 3$	$a =$	(numerical)
	$d \geq 4$	$a = d - 2$	(proven for $d > 4$)

Because of the behaviour for $d \geq 4$, one usually writes $a = d - 2 + \eta$, and one calls η the "anomalous exponent". We will explain the exponent $d - 2$ below.

Rigorous results include: β_c is unique, $\xi < \infty$ up to β_c $\Lambda \nearrow \mathbf{Z}^d$ limit exists $\forall \beta, h$, and much else. The deepest is $\eta = 0$ for $d > 4$.

There are two different kinds of phase transitions in the Ising model:

1. First order transition Let $m(h, \beta) = \langle \sigma_x \rangle$ [for $h \neq 0$ $\langle - \rangle$ is unique and translation invariant (see below), so $\langle \sigma_x \rangle$ is independent of x].

If $\beta > \beta_c$ (low temperature) then

PICTURE

I.e. $\lim_{h \downarrow 0} m(h, \beta) = - \lim_{h \uparrow 0} m(h, \beta) \neq 0$
i.e. m is *discontinuous* at $h = 0$ (if $\beta < \beta_c, m$ is continuous).

2. Second order transition Let $h = 0$, consider

$$\langle \sigma_0 \rangle_\beta^+ \left(= \lim_{h \downarrow 0} \langle \sigma_0 \rangle_{\beta, h} \right) \equiv m(\beta).$$

Then $m(\beta) > 0$ for $\beta > \beta_c$, $m(\beta) = 0$ for $\beta < \beta_c$. What about $m(\beta_c)$? m is *continuous* at β_c. Let us use $T = \beta^{-1}$ as a parameter.

PICTURE

i.e. $\lim_{T\nearrow T_c} m(T) = 0 = m(T_c)$.
However, $\mid \frac{dm}{dT} \mid\to \infty$ as $T \nearrow T_c$. One has $m(T) \sim |T_c - T|^\beta$ β (not T^{-1}!) is a critical exponent for the magnetization.

in $d = 2$, $\beta = \frac{1}{8}$,
for the $d = 3$, $\beta = 0.312 \pm 0.003$
in $d \geq 4$ one expects $\beta = \frac{1}{2}$ (proven for $d \geq 5$).

The continuity or discontinuity of the *order parameter* (m here) is refered to as second order and first order transition. Also: ξ remains finite at a first order transition but becomes infinite at second order (true in Ising).

Remark 1 The states above are *translation invariant* i.e.

$$\langle\sigma_A\rangle = \langle\sigma_{A+x}\rangle \qquad \forall A \subset \mathbf{Z}^d, \forall x \in \mathbf{Z}^d$$

where $A + x = \{a + x | a \in A\}$. Thus Ising has 1 or 2 translation invariant states. There are also *non-translation invariant ones.* One considers *b.c.*:

PICTURE

This forces a contour. In $d = 2$ one does not get a new state, but in $d = 3$ one does get one with an *interface* : $\langle\sigma_x\rangle > 0$, x above, $\langle\sigma_x\rangle < 0$, x below.

Remark 2 Why is $h \neq 0$ unique? Let e.g. $h > 0$ Let us try to construct a state:

PICTURE

The weight of $\sigma = -1$ is $e^{+\beta(\sum\sigma_x\sigma_y + h\sum\sigma_x)} = e^{\beta|\bar{\mathcal{B}}_\Lambda|}e^{-\beta h|\Lambda|}$.
The weight of the configuration where we have a contour $\gamma = \partial\Lambda$:

PICTURE

$= e^{\beta|\bar{\mathcal{B}}_\Lambda|}e^{-2\beta|\partial\Lambda|}$. For Λ a $L-$box $|\partial\Lambda| \sim L^{d-1}, |\Lambda| \sim L^d$ so the contour is more probable! Hence, we expect that the $\Lambda \to \mathbf{Z}^d$ limit is the same as that of $+$ *b.c.* One can make a proof along these lines.

Remark 3 What about h which is not constant :

$$\beta\mathcal{H} = \beta[\sum \sigma_x\sigma_y + \sum_{x\in\Lambda} h_x\sigma_x].$$

Let us consider h_x's *random*: let each h_x be a random variable with zero mean : $\overline{h_x} = 0$ and different h_x, h_y are independent, identically distributed. What do we expect? Consider low temperature and say + boundary conditions. A contour γ costs now an energy and it weight is

$$\exp(-\beta[|\gamma| + 2\sum_{x\in\mathrm{Int}\gamma} h_x)$$

PICTURE

Let γ be an L-cube, so $|\gamma| \propto L^{d-1}$, how about $\sum_{x\in\mathrm{Int}\,\gamma} h_x$?

This is a sum of L^d independent random variables, therefore of size $\propto L^{d/2}$

$$\left(\overline{(\sum h_x)^2} = \sum_{x,y}\overline{h_x h_y} = \sum_x \overline{h_x^2} + \sum_{x\neq y}\bar{h}_x\bar{h}_y = L^d\overline{h^2},\ \text{so}\ \sqrt{\overline{(\sum h_x)^2}} \sim L^{d/2}\right).$$

Thus, if $d-1 > \frac{d}{2}$ or $d > 2$, the magnetic field is unlikely to suppress contours and we expect 2 states. For $d \leq 2$ it seems to have a chance.

Theorem. *For β large, $\overline{h^2}$ small, $d \geq 3$ our model has 2 states $\langle\sigma_x\rangle > 0$ or < 0. For $d \leq 2$ only 1. [This holds with Probability one in h].*

4.6 Spaces of Hamiltonians

Let us say a few words about the general formalism of classical statistical mechanics on a lattice.

1. The "spins"σ_x can take more general values then ± 1. By a *bounded spin model* one means $\sigma_x \in \mathcal{M}$ where $\mathcal{M}$ is "bounded", generally a compact metric space. Examples:

a) Potts models $\sigma_x \in \{1, 2, \cdots, n\}$,

$$\mathcal{H}(\sigma) = -\sum_{|x-y|=1}\delta_{\sigma_x\sigma_y}$$

(δ is Kronecker delta). Thus minimum energy: nearest neighbours want to be the same. For $n = 2$ this is Ising : $\delta_{\sigma\sigma'} = \frac{1}{2}(1+\sigma\sigma')$. The model has S_n symmetry (= group of permutations on n objects): Let $\pi \in S_n$ (so π is a bijection from $\{1, \cdots, n\}$ into itself), and $T_\pi : \Omega_\Lambda \to \Omega_\Lambda : (T_\pi\sigma)_x = \pi\sigma_x$. Then $\mathcal{H}(T_\pi\sigma) = \mathcal{H}(\sigma)$. At low temperatures we have n phases. There is β_c and the transition is *first order* for n large. In particular, one expects $\xi < \infty$, $\forall\beta$, which is proven for n large.

b) $O(N)$-models. Here $\sigma_x \in S^{N-1}$ = sphere in $\mathbf{R}^N$ i.e. σ_x is a vector $\vec{\sigma}_x \in \mathbf{R}^N \|\vec{\sigma}_x\|^2 = 1$. We take

$$\mathcal{H} = -\sum_{|x-y|=1} \vec{\sigma}_x . \vec{\sigma}_y .$$

This has $O(N)$ symmetry : Let $R \in O(N)$ i.e. $R = N \times N$ orthogonal matrix, $R^T R = \mathbb{1}$. Then $\mathcal{H}(R\vec{\sigma}) = \mathcal{H}(\vec{\sigma})$ where $(R\vec{\sigma})_x = R\vec{\sigma}_x$. The $N = 2$ case is called the XY-model, $N = 3$ the classical Heisenberg model. These are very interesting. First, there is no symmetry breaking when $d = 2$: there are no states with $\langle \vec{\sigma}_x \rangle \neq 0$. This is called the *Mermin-Wagner theorem*. We'll prove it later.

In $d \geq 3$ the $O(N)$ models have symmetry breaking :

$$\text{for } \beta \leq \beta_c,\ \langle \vec{\sigma}_x \rangle = 0, \quad \text{for } \beta > \beta_c \ \ \langle \vec{\sigma}_x \rangle = m(\beta)\hat{n}, \quad \hat{n} \in S^{N-1}.$$

There are infinitely many low-temperature states, parametrized by unit vectors $\hat{n}$.

For $d = 2$, $N = 2$ and $N \geq 3$ behave qualitatively differently. For $N = 2$ (XY-model or *plane rotator*), there is a critical temperature β_c where ξ becomes ∞: For $\beta < \beta_c$, $\xi(\beta) < \infty, \beta \geq \beta_c$, $\xi(\beta) = \infty$ and the correlation function has a peculiar decay if $\beta > \beta_c$: $\langle \sigma_x \sigma_y \rangle \sim |x-y|^{-a(\beta)} a(\beta)$ depends on β.

For $d = 2, N \geq 3$, one expects $\xi(\beta) \sim e^{+c\beta}$ i.e. $\xi \to \infty$ as $\beta \to \infty$ (which is proven as a lower bound). We will return to this with the Renormalization Group.

c) <u>Gauge theories</u>
Consider bounds b with *orientation* $b = (x, y) \neq (y, x)$ and denote $(y, x) = b^{-1}$.
PICTURE
Configuration space : spins $\sigma_b \in SU(N)$ ($N \times N$ unitary matrices, det $\sigma_b = 1$). Let $\sigma_{b^{-1}} = (\sigma_b)^{-1} \cdot \Omega_\Lambda = (SU(N))^{\mathcal{B}_\Lambda}$. Let p be a plaquette and choose arbitarily orientation:

PICTURE

Put :

$$\begin{aligned} S_p = \text{Re Tr } \sigma_{b_1}\sigma_{b_2}\sigma_{b_3}\sigma_{b_4} &= \frac{1}{2}(\text{Tr } \sigma_{b_1}\sigma_{b_2}\sigma_{b_3}\sigma_{b_4} + \overline{\text{Tr } \sigma_{b_1} \cdots \sigma_{b_4}}) \\ &= \frac{1}{2}(\text{Tr } \sigma_{b_1} \cdots \sigma_{b_4} + \text{Tr } (\sigma_{b_1} \cdots \sigma_{b_4})^{-1}) \end{aligned}$$

(since $\overline{\text{Tr } U} = \text{Tr } U^+ = \text{Tr } U^{-1}$ for $U \in SU(N)$) and

$$\mathcal{H}_\Lambda = \sum_{p c \Lambda} S_p .$$

This Hamiltonian has a huge symmetry, the *gauge symmetry* : Let $G_\Lambda = \times_{x\in\Lambda} SU(N)$ i.e. $g \in G_\Lambda$ is a map $\Lambda \ni x \to g_x \in SU(N)$ i.e. at each point $x \in \Lambda$ choose a matrix $g_x \in SU(N)$. G_Λ is the *gauge group* in Λ.
For $\sigma \in \Omega_\Lambda, g \in G_\Lambda$, let

$$(U(g)\sigma)_b = g_x \sigma_b g_y^{-1} \text{ if } b = (x, y)$$

Then

PICTURE

$$S_p(U(g)\sigma) = \text{Re Tr } [g_{x_1}\sigma_{b_1} g_{x_2}^{-1} \cdot g_{x_2}\sigma_{b_2} g_{x_3}^{-1} \cdots g_{x_4}^{-1} g_{x_4}\sigma_{b_4} g_{x_1}^{-1}] = S_p(\sigma)$$

so

$$\mathcal{H}_\Lambda(U(g)\sigma) = \mathcal{H}_\Lambda(\sigma).$$

This is a big symmetry and in all d remains *unbroken*.

2. The last example had a 4-spin interaction. Let us now generalize to give the general formalism of bounded spin models. Thus, let $\mathcal{M}$ be compact metric space, the infinite configuration space $\Omega = \mathcal{M}^{\mathbf{Z}^d}$, finite volume $\Omega_\Lambda = \mathcal{M}^\Lambda$.
Ω is still a compact metric space (Tychonov), with, say, metric $D(\sigma, \sigma') = \sum_{x\in\mathbf{Z}^d} 2^{-|x|} d(\sigma_x, \sigma'_x)$, d a metric on $\mathcal{M}$. The measures on Ω we are interested in are characterized by two data: a) Hamiltonian b) a priori measure:

a) Hamiltonian We want to have general interactions.

Definition A potential Φ is a collection of continuous functions

$$\Phi_X : \mathcal{M}^X \to \mathbf{R} \quad \text{for } X \subset \mathbf{Z}^d, \quad |X| < \infty.$$

Example Ising : $\Phi_X(\sigma) = \begin{cases} \sigma_x\sigma_y & \text{if } X = \{x, y\} = \text{bond} \\ 0 & \text{otherwise} \end{cases}$

Hamiltonian in Λ , $|\Lambda| < \infty$ is $\mathcal{H}_\Lambda(\sigma) = \sum_{X\subset\Lambda} \Phi_X(\sigma)$ (*free boundary condition*).
(compare with previous models ! in gauge theories we rather consider spins on bonds, not sites).

b) A priori measure: Let ν be a Borel probability measure on $\mathcal{M}$ (Examples: Ising : $\nu(-1) = \nu(1) = \frac{1}{2}$.
$\mathcal{O}(N)$-model ν is uniform measure on S^{N-1} i.e. in $\mathbf{R}^N$ $d\nu(\vec{x}\,) = c\delta(\vec{x}^{\,2} - 1)d^N x$, c^{-1} = volume of S^{N-1}.
Gauge theories : ν is the Haar measure on $SU(N)$.

Definition Gibbs measure in volume Λ, with free boundary conditions, is

$$d\mu_\Lambda(\sigma) = \frac{1}{Z_\Lambda} e^{-\beta\mathcal{H}_\Lambda(\sigma)} \prod_{x\in\Lambda} d\nu(\sigma_x) \tag{4.14}$$

Boundary conditions Let $\bar\sigma \in \Omega$. Put, for $\sigma \in \Omega_\Lambda$,

$$\mathcal{H}^{\bar\sigma}_\Lambda(\sigma) = \sum_{X:X\cap\Lambda\neq\emptyset} \Phi_X(\sigma \vee \bar\sigma) \tag{4.15}$$

where use the notation : $\sigma \vee \bar\sigma \in \Omega : (\sigma\vee\bar\sigma)_x = \begin{cases} \sigma_x & x\in\Lambda \\ \bar\sigma_x & x \in \Lambda^c \end{cases}$

Example In Ising we add so $\mathcal{H}^{free}$, the terms

$$\mathcal{H}^{\bar\sigma} = \mathcal{H}^{free} + \sum_{\substack{x\in\Lambda,\, y\in\Lambda^c \\ |x-y|=1}} \sigma_x\bar\sigma_y .$$

Note that (4.15) is an infinite sum so we need to address convergence.

Definition
a) Φ is *finite range* if $\exists R : \Phi_X = 0$ for all X such that diameter of X is $> R$. (diam $X = \max_{x,y\in X} |x-y|$). For finite range Φ, (4.15) is finite trivially, since continuity of Φ_X in $\sigma \Rightarrow \|\Phi_X\| = \sup_{\sigma\in\mathcal{M}^X} |\Phi_X(\sigma)| < \infty$ ($\mathcal{M}^X$ is compact).

Note Φ is *translation invariant* if $\Phi_X = \Phi_{X+a}$ $\forall a \in \mathbf{Z}^d$ where $X+a = \{x+a | x\in X\}$. Let us assume so.
b) Sometimes finite range is not general enough. There is a convenient class of potentials, that allows for the $\Lambda \nearrow \mathbf{Z}^d$ limit of the free energy, the convergence of high temperature expansions etc. Let us denote

$$\|\Phi\| = \sup_{\sigma\in\Omega_X} |\Phi_X(\sigma)|$$

and consider the set $\mathcal{B}$ of Φ's : $\Phi \in \mathcal{B}$ if

$$\|\Phi\| = \sum_{\substack{X\subset \mathbf{Z}^d \\ 0\in X}} \|\Phi_X\| < \infty$$

i.e. if the interaction energy of σ_0 with all other spins is finite. Note that translation invariance $\Rightarrow \|\Phi\| = \sum_{\substack{X\subset\mathbf{Z}^d \\ y\in X}} \|\Phi_X\|$ for any $y \in \mathbf{Z}^d$.
$\mathcal{B}$ is a vector space, $\|\ \|$ is a norm and $\mathcal{B}$ is *complete* in this norm, i.e. if $\|\Phi^{(n)} - \Phi^{(m)}\| \xrightarrow[n,m\to\infty]{} 0$ then $\exists\Phi\in\mathcal{B}$ such that $\|\Phi - \Phi^{(n)}\| \to 0$ as $n\to\infty$. So $\mathcal{B}$ is a Banach space.

Lemma Let $\Phi \in \mathcal{B}$. Then

$$|\mathcal{H}_\Lambda^{\bar\sigma}(\sigma)| \le |\Lambda| \; \|\Phi\|.$$

Proof We have

$$|\sum_{X:X\cap\Lambda\neq\emptyset} \Phi_X(\sigma)| = |\sum_{y\in\Lambda} \sum_{\substack{X\subset \mathbf{Z}^d \\ y\in X}} \frac{1}{|X\cap\Lambda|}\Phi_X(\sigma)| \le \sum_{y\in\Lambda} \sum_{\substack{X\subset \mathbf{Z}^d \\ y\in X}} \|\Phi_X\| = |\Lambda| \; \|\Phi\|$$

Now, for $\Phi \in \mathcal{B}$, define the *Gibbs measure on* Ω_Λ *with b.c.* $\bar\sigma \in \Omega$:

$$d\mu_\Lambda^{\bar\sigma}(\sigma) = \frac{1}{Z_\Lambda^{\bar\sigma}} e^{-\beta\mathcal{H}_\Lambda^{\bar\sigma}(\sigma)} d\nu_\Lambda(\sigma) \tag{4.16}$$

where $\sigma \in \Omega_\Lambda$,

$$\begin{aligned} d\nu_\Lambda(\sigma) &= \prod_{x\in\Lambda} d\nu(\sigma_x) \\ Z_\Lambda^{\bar\sigma} &= \int_{\Omega_\Lambda} e^{-\beta\mathcal{H}_\Lambda^{\bar\sigma}(\sigma)} d\nu_\Lambda(\sigma) \end{aligned}$$

We have

Theorem. *Let* $\Phi \in \mathcal{B}$. *Then the free energy*

$$F = \lim_{L\to\infty} \frac{1}{L^d} \log Z_{\Lambda_L}^{\bar\sigma} \qquad (\Lambda_L = L - \text{box})$$

exists and is independent of $\bar\sigma$.

Proof

a) Independence of $\bar\sigma$. Consider

$$|\mathcal{H}_\Lambda^{\bar\sigma}(\sigma) - \mathcal{H}_\Lambda^{\bar\sigma'}(\sigma)| \le \sum_{X\cap\Lambda\neq\emptyset, X\cap\Lambda^c\neq\emptyset} |\Phi_X(\sigma\vee\bar\sigma) - \Phi_X(\sigma\vee\bar\sigma')|$$

(since terms with $X \subset \Lambda$ cancel)

$$\begin{aligned} &= \sum_{y\in\Lambda} \sum_{\substack{X \text{as above} \\ y\in X}} \frac{1}{|X\cap\Lambda|} |\Phi_X(\sigma\vee\bar\sigma) - \Phi_X(\sigma\vee\bar\sigma')| \\ &\le 2\sum_{y\in\Lambda} \sum_{\substack{X \text{as above} \\ y\in X}} \frac{1}{|X\cap\Lambda|} \|\Phi_X\| \end{aligned}$$

Divide $\sum_y$ into

1° dist $(y, \Lambda^c) \le L_0$

$2°$ dist $(y,\Lambda^c)\geq L_0$

PICTURE

Then

$1°\leq CL_0L^{d-1}\|\Phi\|$

$2°\leq 2L^d\frac{1}{L_0}\|\Phi\|$

$\uparrow$

$|X\cap\Lambda|\geq L_0$

Thus

$$Z_\Lambda^{\bar\sigma'}e^{C_2|\Lambda|[\frac{L_0}{L}+\frac{1}{L_0}]}\leq Z_\Lambda^{\bar\sigma}\leq Z_\Lambda^{\bar\sigma'}e^{C_1|\Lambda|[\frac{L_0}{L}+\frac{1}{L_0}]}$$
$$\Rightarrow \lim_{L\to\infty}\frac{1}{|\Lambda_L|}\log Z_{\Lambda_L}^{\bar\sigma'}+\frac{C_2}{L_0}\leq \lim_{L\to\infty}\frac{1}{|\Lambda_L|}\log Z_\Lambda^{\bar\sigma}\leq \lim_{L\to\infty}\frac{1}{|\Lambda_L|}\log Z_{\Lambda_L}^{\bar\sigma'}+\frac{C_1}{L_0}$$

for all L_0, so $\lim_{L\to\infty}\frac{1}{|\Lambda_L|}\log Z_{\Lambda_L}^{\bar\sigma'}=\lim_{L\to\infty}\frac{1}{|\Lambda_L|}\log Z_{\Lambda_L}^{\bar\sigma}$

b) Existence

1. Let $\epsilon>0$. Then $\exists R$ such that $\|\Phi-\Phi^{(R)}\|\leq\epsilon$ when $\Phi_X^{(R)}=\begin{cases}\Phi_X & \text{diam }(X)<R\\ 0 & \text{diam }(X)\geq R\end{cases}$.

So

$$|\mathcal{H}_{\Lambda,\Phi}^{\bar\sigma}-\mathcal{H}_{\Lambda,\Phi^{(R)}}^{\bar\sigma}|\leq\epsilon|\Lambda|$$
$$\Rightarrow |\frac{1}{|\Lambda|}\log Z_{\Lambda,\Phi}^{\bar\sigma}-\frac{1}{|\Lambda|}\log Z_{\Lambda,\Phi^{(R)}}^{\bar\sigma}|\leq\epsilon.$$

Thus, it suffices to consider Φ of finite range, say R.

2. For Φ of finite range, pick $L>L_0$, $\frac{L}{L_0}\in\mathbf{N}$. Cover Λ_L with $(\frac{L}{L_0})^d$ L_0-boxes: call them $\Lambda^{(i)}$ call $\sigma|_{\Lambda^{(i)}}\equiv\sigma^{(i)}$. Get, as above,

$$|\mathcal{H}_{\Lambda_L}^{\bar\sigma}(\sigma)-\sum_{i=1}^{(L/L_0)^d}\mathcal{H}_{\Lambda_{L_0}}^{\bar\sigma}(\sigma^{(i)})|\leq CL_0^{d-1}(\frac{L}{L_0})^d$$

where C depends on Φ,R. Thus,

$$\begin{aligned}\int e^{-\beta\sum\mathcal{H}_{\Lambda_{L_0}}^{\bar\sigma}(\sigma^{(i)})}d\nu_{\Lambda_L}e^{-C_2L_0^{-1}L^d} &\leq \int e^{-\beta\mathcal{H}_{\Lambda_L}^{\bar\sigma}(\sigma)}d\nu_{\Lambda_L}\\ &\leq \int e^{-\beta\sum\mathcal{H}_{\Lambda_{L_0}}^{\bar\sigma}(\sigma^{(i)})}d\nu_{\Lambda_L}e^{C_1L_0^{-1}L^d}\end{aligned}\tag{4.17}$$

and since $\int e^{-\beta\sum\mathcal{H}_{\Lambda_{L_0}}^{\bar\sigma}(\sigma^{(i)})}d\nu_{\Lambda_L}=(Z_{\Lambda_{L_0}}^{\bar\sigma})^{(L/L_0)^d}$ (use $d\nu_{\Lambda_L}(\sigma)=\prod_{i=1}^{(L/L_0)^d}d\nu_{\Lambda_{L_0}}(\sigma^{(i)})$) to get

$$|\frac{1}{L^d}\log Z_{\Lambda_L}^{\bar\sigma}-\frac{1}{L_0^d}\log Z_{\Lambda_{L_0}}^{\bar\sigma}|\leq CL_0^{-1}$$
$$\Rightarrow |\lim_{L\to\infty}\sup\frac{1}{L^d}\log Z_{\Lambda_L}^{\bar\sigma}-\frac{1}{L_0^d}\log Z_{\Lambda_{L_0}}^{\bar\sigma}|\leq CL_0^{-1}$$

$$\Rightarrow |\lim_{L\to\infty}\sup \frac{1}{L^d}\log Z^{\bar\sigma}_{\Lambda_L} - \lim_{L_0\to\infty}\inf \frac{1}{L_0^d}\log Z^{\bar\sigma}_{\Lambda_{L_0}}| \le 0$$
$$\Rightarrow \limsup = \liminf \quad \Rightarrow \quad \text{limit exists}$$

Note 1. Idea in both parts was, that *boundary energy/volume* $\to 0$ as $L \to \infty$.

Note 2. This does not mean that $\langle - \rangle^{\bar\sigma}$ is $\bar\sigma$-independent. Only the free energy is boundary independent.

Remark $\mathcal{B}$ includes Ising type systems

$$\mathcal{H} = \sum_{x,y\in\Lambda} J_{xy}\sigma_x\sigma_y \tag{4.18}$$

provided J_{xy} has decay as $|x-y| \to \infty$: e.g. $|J_{xy}| \le \frac{C}{[|x-y|+1]^{d+\epsilon}}$ will do.
There is a very important system where $\Phi \notin \mathcal{B}$: the Coulomb gas. It is like (4.18), $\sigma_x =$ charge at x i.e. $+$ or $-$ charges and

$$\mathcal{H}_\Lambda(\sigma) = \sum_{\substack{x,\, y\, \in\, \Lambda \\ x\, \neq\, y}} \frac{1}{|x-y|}\sigma_x\sigma_y$$

Hence $\Phi_X = \begin{cases} \frac{1}{|x-y|} & \text{if } X = (x,y) \\ 0 \text{ otherwise} \end{cases}$ and $\sum_{0\in X}\Phi_X = \sum_{y\neq 0}\frac{1}{|y|} = \infty$. Nevertheless, the $\Lambda \nearrow \mathbf{Z}^d$ limit exists. The secret is signs: note that $e^{-\beta\mathcal{H}}$ gets very small contribution from $\sigma_x = +1 \;\; \forall x$ or $\sigma_x = -1 \;\; \forall x$. The nonzero contribution comes from alternating configurations (neutral ones: σ is charge). See Simon, p. 121.

The High-temperature analysis goes through for $\Phi \in \mathcal{B}$. One expands

$$e^{-\beta\mathcal{H}_\Lambda} = \prod_X e^{-\beta\Phi_X} = \prod_X e^{-\beta\|\Phi_X\|}\prod_X e^{\beta\tilde\Phi_X} \tag{4.19}$$

$(\tilde\Phi_X = \|\Phi_X\| - \Phi_X \ge 0)$

$$\prod_X e^{+\beta\tilde\Phi_X} = \prod_X(1+f_X) = \sum_{\{X_\alpha\}}\prod_\alpha f_{X_\alpha}$$

$(f_X = e^{\beta\tilde\Phi_X} - 1 \ge 0)$, and one proceeds as before to prove $|\langle\sigma_A\sigma_B\rangle - \langle\sigma_A\rangle\langle\sigma_B\rangle| \to 0$ as dist $(A,B) \to \infty$. See BK.

5 Gibbs States and DLR Equations.

Finally, let us define what one means by Gibbs states in the general framework.
Recall the discussion of correlations converging $\Rightarrow \exists$ infinite volume measure. That can be repeated in the general framework. Let us state things slightly differently. Given $\Phi \in \mathcal{B}$, an a priori measure ν on $\mathcal{M}$ and $\bar{\sigma} \in \Omega$ recall (4.16) above, the Gibbs measure $\mu_\Lambda^{\bar{\sigma}}$ on Ω_Λ. We can extend $\mu_\Lambda^{\bar{\sigma}}$ to Ω by letting $\delta_{\Lambda^c}^{\bar{\sigma}}$ be the Dirac measure on Ω_{Λ^c} i.e. $\int f(\tau) d\delta_{\Lambda^c}^{\bar{\sigma}}(\tau) = f(\bar{\sigma}_{\Lambda^c})$ $\forall f$ continuous on Ω_{Λ^c} (here $\bar{\sigma}_{\Lambda^c}$ means just the restriction of the configuration $\bar{\sigma}$ to Λ^c, where f is defined). Put $\mu^{\bar{\sigma},\Lambda}$ (measure on $\Omega = \Omega_\Lambda \times \Omega_{\Lambda^c}$) $= \mu_\Lambda^{\bar{\sigma}}$ (on Ω_Λ). $\delta_{\Lambda^c}^{\bar{\sigma}}$ (on Ω_{Λ^c}).

Thus: If $g \in C(\Omega_A)$ $A \subset \Lambda$, then $\int g d\mu^{\bar{\sigma},\Lambda} = \int g d\mu_\Lambda^{\bar{\sigma}} \equiv \langle g \rangle_\Lambda^{\bar{\sigma}}$ our old state on $C(\Omega_\Lambda)$.

Let $B(\Omega) = \{$ Borel probability measures on $\Omega\}$. We say $\mu_n \in B(\Omega)$ converge if $\int f d\mu_n$ converge $\forall f \in C(\Omega) =$ continuous functions on Ω.

Fact $B(\Omega)$ is compact in this topology i.e. if $\mu_n \in B(\Omega)$ $n \in \mathbf{N}$ then there is a convergent subsequence and limit $\mu \in B(\Omega) : \mu_{n_i} \longrightarrow_{i \to \infty} \mu$. Hence, there is a sequence of boxes $\Lambda_n \nearrow \mathbf{Z}^d$ such that $\mu^{\bar{\sigma},\Lambda_n}$ converges to some measure μ. We call the *set of all such limit points*, with *all possible* $\bar{\sigma}$, *the set of Gibbs measures* $\mathcal{G}_\Phi$ of the potential Φ.

Theorem. $\mu \in \mathcal{G}_\Phi$ if and only if $\forall \Lambda \subset \mathbf{Z}^d, |\Lambda| < \infty$ $\exists$ probability measure $\tilde{\mu}_{\Lambda^c}$ on Ω_{Λ^c} such that $\forall f \in C(\Omega_\Lambda)$

$$\int f(\sigma) d\mu(\sigma) = \int \left(\int f(\sigma_\Lambda) d\mu_\Lambda^{\sigma_{\Lambda^c}}(\sigma_\Lambda) \right) \quad d\tilde{\mu}_{\Lambda^c}(\sigma_{\Lambda^c}) \ \ (DLR) \tag{5.1}$$

Remark The first integral is the integral of f w.r.t. the Gibbs measure in finite volume, with b.c. σ_{Λ^c}, and (5.1) is some average over boundary conditions. (DLR) is the *DLR-equation* (Dobrushin, Lanford, Ruelle).

Proof "$\Rightarrow$"Let $f \in C(\Omega_\Lambda)$. So $\int f d\mu = \lim_{n\to\infty} \int f d\mu^{\bar{\sigma},\Lambda_n}$ for some $\bar{\sigma}$, some $\Lambda_n \nearrow \mathbf{Z}^d$ $n \to \infty$. Explicitely,

$$\int f d\mu^{\bar{\sigma},\Lambda_n} = \int f(\sigma_\Lambda) e^{-\beta \mathcal{H}_{\Lambda_n}^{\bar{\sigma}}(\sigma)} d\nu_{\Lambda_n}(\sigma) / \int e^{-\beta \mathcal{H}_{\Lambda_n}^{\bar{\sigma}}(\sigma)} d\nu_{\Lambda_n} \tag{5.2}$$

Write, with $\tilde{\sigma}_x = \sigma_x$, $x \in \Lambda_n$, $= \bar{\sigma}_x$, $x \in \Lambda_n^c$,

$$\mathcal{H}_{\Lambda_n}^{\bar{\sigma}}(\sigma) = \sum_{X \cap \Lambda_n \neq \emptyset} \Phi_X(\tilde{\sigma}) = \mathcal{H}_\Lambda^{\tilde{\sigma}}(\sigma) + \sum_{\substack{X \subset \Lambda^c \\ X \cap \Lambda_n \neq \emptyset}} \Phi_X(\tilde{\sigma}) \equiv \mathcal{H}_\Lambda^{\tilde{\sigma}} + \tilde{\mathcal{H}}$$

where the second sum is independent of σ_Λ.

PICTURE

Thus, the numerator of (5.2) equals

$$\int\Big(\int f(\sigma_\Lambda)e^{-\beta\mathcal{H}_\Lambda^{\tilde\sigma}(\sigma_\Lambda)}d\nu_\Lambda(\sigma_\Lambda)\Big)e^{-\beta\tilde{\mathcal{H}}}d\nu_{\Lambda_n\backslash\Lambda}(\sigma)$$
$$= \int(\int fd\mu_\Lambda^{\tilde\sigma})Z_\Lambda^{\tilde\sigma}e^{-\beta\tilde{\mathcal{H}}}d\nu_{\Lambda_n\backslash\Lambda} \tag{5.3}$$

But $\int fd\mu_\Lambda^{\tilde\sigma} = g_n(\sigma_{\Lambda_n\backslash\Lambda}), g_n \in C(\Omega_{\Lambda_n\backslash\Lambda})$ and since

$$Z_\Lambda^{\tilde\sigma} = \int e^{-\beta\mathcal{H}_\Lambda^{\tilde\sigma_\Lambda}(\sigma_\Lambda)}d\nu_\Lambda(\sigma_\Lambda)$$

(5.3) becomes

$$(5.3) = \int g_n(\sigma_{\Lambda_n\backslash\Lambda})e^{-\beta\mathcal{H}_{\Lambda_n}^{\tilde\sigma}(\sigma_{\Lambda_n})}d\nu_{\Lambda_n}(\sigma_{\Lambda_n})$$

and so

$$\int fd\mu^{\bar\sigma,\Lambda_n} = \int g_n(\sigma_{\Lambda_n\backslash\Lambda})d\mu_{\Lambda_n}^{\bar\sigma}(\sigma_{\Lambda_n})$$

Now $g_n \in C(\Omega_{\Lambda_n\backslash\Lambda}) \subset C(\Omega)$ and

$$\sup_{\sigma\in\Omega}|g_n(\sigma)-g(\sigma)| \underset{n\to\infty}{\longrightarrow} 0 \qquad \text{(Prove !)}$$

with $g(\sigma_{\Lambda^c}) = \int f(\sigma_\Lambda)d\mu_\Lambda^{\sigma_{\Lambda^c}}(\sigma_\Lambda)$. Also, $\int gd\mu^{\bar\sigma,\Lambda_n} \to \int gd\mu$ by assumption since $g \in C(\Omega)$. Thus

$$\int fd\mu^{\bar\sigma,\Lambda_n} \to \int gd\mu$$

where $g(\sigma_{\Lambda^c}) = \int f(\sigma_\Lambda)d\mu_\Lambda^{\sigma_{\Lambda^c}}(\sigma_\Lambda)$ as claimed. the Converse requires the concept of conditional expectation and we skip it, see Simon's book .

Let us finish this section by proving the Mermin-Wagner theorem for the XY-model. This generalizes to general systems with compact Lie group symmetries. See Simon's book.

Theorem. Let μ be a (translation invariant) Gibbs measure for the XY-model. Then μ is invariant under a global rotation of the spins. I.e. for any $f \in C(\Omega)$ and $R \in SO(2)$,

$$\int f(\sigma)d\mu(\sigma) = \int f(R\sigma)d\mu(\sigma)$$

where $(R\sigma)_x \equiv R\sigma_x$.

Proof 1. Idea is the following. Consider say $f(\sigma) = F(\sigma_0)$, depends on σ_0 only. Let $R(\phi)$ be rotation by angle $\phi, R(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$. Go to angular variables $\sigma_x =$

$\begin{pmatrix} \cos_x \\ \sin_x \end{pmatrix}$, so $\mathcal{H} = -\sum_{\langle xy \rangle} \cos(_x - _y)$, a'priori measure is $\frac{d_x}{2\pi}$ on $[0, 2\pi]$.
We want to prove :

$$\langle F(\theta_0 + \phi) \rangle = \langle F(_0) \rangle \quad \forall \phi$$

i.e. suffices to prove $\langle \frac{d}{d\phi}|_{\phi=0} F(_0 + \phi) \rangle = 0 \ \ \forall F$ smooth. Now $\langle - \rangle$ is approximatively $\langle - \rangle^{\bar{}}{}_\Lambda$ some large Λ, some b.c. $\bar{}$. Thus we could change variables

$$\langle F(_0 + \phi) \rangle^{\bar{}}{}_{\Lambda = \langle F(_0) \rangle^{+\bar{}\phi_\Lambda}}$$

which followed since $\mathcal{H}^{\bar{}}{}_{\Lambda(-\phi) = \mathcal{H}^{+\bar{}\phi_\Lambda()}}$

$$\left(\mathcal{H}^{\bar{}}{}_{\Lambda() = -\sum_{x,y \in \Lambda} \cos(_x - _y) - \sum_{\substack{x \in \Lambda \\ y \in \Lambda^c}} \cos(_x - _y)} \right)$$

So ϕ dependence is in the b.c. of course. This is tricky to control, so let us try to change variables by slowly rotating the spins : rotate $\theta_0 + \varphi \to \theta_0$ and the rest a bit less such that on $\partial\Lambda$ there is no rotation: Let $g : \Lambda \to \mathbf{R}$ $g(0) = 1, g(x) = 0$ for x near $\partial\Lambda$. Then, let $(\tau_\varphi \theta)_x = \theta_x + g(x)\varphi$

$$\begin{aligned}
&\langle F(\theta_0 + \varphi) \rangle^{\bar{\sigma}}_\Lambda = \langle F((\tau_\varphi \theta)_0) \rangle^{\bar{\sigma}}_\Lambda \\
&= \frac{1}{Z^{\bar{\sigma}}_\Lambda} \int F(\theta_0) \exp[-\beta \mathcal{H}^{\bar{\sigma}}_\Lambda(\tau_\varphi \theta)] \prod \frac{d\theta_x}{2r} \\
&\uparrow \\
&\theta_x \to \theta_x - g(x)\phi = (\tau_{-\phi}\theta)_x \\
&\Rightarrow \langle \frac{d}{d\phi}|_0 F(\theta_0 + \phi) \rangle^{\bar{\sigma}}_\Lambda = -\beta \langle (\frac{d}{d\phi}|_0 \mathcal{H}^{\bar{\sigma}}_\Lambda(\tau_\varphi \theta)) F(\theta_0) \rangle^{\bar{\sigma}}_\Lambda
\end{aligned}$$

Thus, by Schwartz inequality:

$$|\langle \frac{d}{d\phi}|_0 F \rangle| \ \ \leq \beta \langle F^2 \rangle^{1/2} \langle (\frac{d}{d\phi}|_0 \mathcal{H}^{\bar{\sigma}}_\Lambda)^2 \rangle^{1/2}$$

and using the above argument backwards,

$$\beta \langle (\frac{d}{d\phi}|_0 \mathcal{H}^{\bar{\sigma}}_\Lambda)^2 \rangle = \langle \frac{d^2}{d\phi^2}|_0 \mathcal{H}^{\bar{\sigma}}_\Lambda(\tau_{-\phi}\theta) \rangle$$

But

$$\frac{d^2}{d\phi^2} \mathcal{H}^{\bar{\sigma}}_\Lambda(\tau_{-\phi}\theta) = \frac{d^2}{d\phi^2} \sum_{\langle xy \rangle x \in \Lambda, y \in \Lambda} \cos(\theta_x - \theta_y - \phi(g(x) - g(y))$$

where we sum only over $x, y \in \Lambda$ since $g(x) = 0$ near $\partial\Lambda$.
So

$$\begin{aligned}
&(\frac{d^2}{d\phi^2}|_\phi \mathcal{H}^{\bar{\sigma}}_\Lambda) = |\sum (g(x) - g(y))^2 \cos(\theta_x - \theta_y)| \\
&\leq \sum_{\langle xy \rangle \text{ in } \Lambda} (g(x) - g(y))^2
\end{aligned}$$

Thus,

$$|\langle \frac{d}{d\phi}|_0 F\rangle| \le C \sum_{\substack{\langle xy\rangle \\ x,\, y\, \in\, \Lambda}} (g(x)-g(y))^2 \tag{5.4}$$

where C is g-independent. (5.4) holds for all $g:\Lambda\to\mathbf{R}, g(0)=1, g=0$ near $\partial\Lambda$. Also (5.4) holds for any Λ (C is Λ-independent).
To make the above argument rigorous, we use the DLR-equation to write

$$\int \frac{dF}{d\phi} d\mu = \int [\int \frac{dF}{d\phi} d\mu_\Lambda^{\bar\sigma}(\sigma)] d\tilde\mu_{\Lambda^c}(\bar\sigma)$$

i.e. an average over $\bar\sigma$ of what we did. Then (5.4) follows as above.

Proof 2. Second idea. The point now is that

$$\inf_\Lambda \inf_g \sum_{\langle xy\rangle} (g(x)-g(y))^2 = 0.$$

Reason : we show that this equals

$$\inf_{h\in C_0^\infty, h(0)=1} \int (\nabla h)^2 d^2x \tag{5.5}$$

and this equals zero. Thus, let $h\in C_0^\infty$, $h(0)=1$ (smooth function $h:\mathbf{R}^2\to\mathbf{R}$ with compact support). Let the support of h be contained in Λ_R (= cube, center 0, side R). Put $g_L:\mathbf{Z}^2\to\mathbf{R}$, $g_L(x)=h(\frac{L}{R}x)$ ($x\in\mathbf{Z}^2$). Then $g_L(0)=1$, $g_L=0$ near $\partial\Lambda_L$ and

$$\sum (g_L(x)-g_L(y))^2 = \sum_{x\in\mathbf{Z}^2}\sum_{i=1}^2 (h(\frac{L}{R}x)-h(\frac{L}{R}(x-e_i)))^2$$

(where $e_1=(1,0), e_2=(0,1)$)

$$=\sum_x\sum_i (\frac{R}{L})^2 [\frac{(h(\frac{L}{R}x)-h(\frac{L}{R}(x-e_i))}{R/L}]^2$$

$\underset{L\to\infty}{\longrightarrow} \int (\nabla h(u))^2 d^2u$. This is zero : let $h_\epsilon(x) = \begin{cases} 1 & |x|<1 \\ |x|^{-\epsilon} & |x|\ge 1\end{cases}$. This is piecewise C^1 and not 0 on boundary, but one can approximate this function by C_0^∞ (show !). Now, $\nabla |x|^{-\epsilon} = -\epsilon\frac{\vec x}{|x|^{2+\epsilon}} \Rightarrow \int(\nabla h)^2 = 2\pi\epsilon^2\int_1^\infty r^{-2-2\epsilon} r dr = \pi\epsilon \to 0$, as $\epsilon\to 0$

Further comments
In the general theory of Gibbs states one can now show the following.
Let $\Phi\in\mathcal{B}$ and $\mathcal{G}_\Phi$ be the set of Gibbs states of Φ. Then :

a) $\mathcal{G}_\Phi$ is a convex set : $\mu_1,\mu_2\in\mathcal{G}_\Phi \Rightarrow$

$$s\mu_1 + (1-s)\mu_2 \in \mathcal{G}_\Phi, \quad s\in[0,1].$$

b) $\mathcal{G}_\Phi$ is also compact (in the weak* topology).

c) There is a set of extremal elements $\mu_\alpha \in \mathcal{G}_\phi$ $\alpha \in A$ (index set), (μ_α is extremal, if it cannot be written as a convex combination of two distinct μ's). such that every $\mu \in \mathcal{G}_\Phi$ is uniquely a convex combination of μ_α (i.e., if $|A| < \infty, A = \{\alpha_i\}_{i=1}^N$ then $\exists s_i \in [0,1], \sum_{i=1}^N s_i = 1$ and $\mu = \sum_{i=1}^N s_i \mu_{\alpha_i}$, in general, $\exists$ measure ν on A, $\nu(A) = 1$ and $\mu = \int \mu_\alpha d\nu(\alpha)$).
The measures μ_α are called *pure phases.*

Example Ising model : Pure phases are μ^+ and μ^-, general $\mu = s\mu^+ + (1-s)\mu^-$. For example $\mu^{free} = \mu^{periodic} = \frac{1}{2}(\mu^+ + \mu^-)$ (Why ?).

d) Pure phases have correlations $\to 0$ at infinity.

$$\langle \sigma_A ; \sigma_B \rangle^T = \langle \sigma_A \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle \longrightarrow_{\text{dist }(A,B)\to\infty} 0$$

Note Pure phases are physical, *mixtures* of these are reflecting our ignorance.

6 The Ginzburg-Landau Model

We now consider models where $\sigma_x \in \mathbf{R}$ i.e. is unbounded.

Motivation :

1. Later we see that our previous models naturally give rise to this by a process called coarse-graining (see Le Bellac also).
2. These are related to quantum fields. So let the "spin"now be $\phi : \mathbf{Z}^d \to \mathbf{R}$. Let us consider a simple Hamiltonian

$$\mathcal{H}_\Lambda(\phi) = -\sum_{\langle xy \rangle} \phi_x \phi_y + a \sum_{x\in\Lambda} \phi_x^2 + \lambda \sum_{x\in\Lambda} \phi_x^4 \tag{6.1}$$

which again has the $\phi \to -\phi$ symmetry, and consider the measure

$$\frac{1}{Z_\Lambda} e^{-\beta\mathcal{H}_\Lambda(\phi)} \prod_{x\in\Lambda} d\phi_x \quad \text{on } \mathbf{R}^{|\Lambda|} \tag{6.2}$$

with $Z_\Lambda = \int_{\mathbf{R}^\Lambda} e^{-\beta\mathcal{H}_\Lambda(\phi)} \prod_{x\in\Lambda} d\phi_x$. This integral exists if $\lambda > 0$. This is easy to see as follows:

$$\begin{aligned}\mathcal{H}_\Lambda(\phi) &= \sum_{\langle xy \rangle} (\phi_x - \phi_y)^2 + (-2d + a) \sum_{x\in\Lambda} \phi_y^2 + \lambda \sum \phi_x^4 \\ &\geq (a - 2d) \sum \phi_x^2 + \lambda \sum \phi_x^4\end{aligned}$$

so

$$Z \leq \left[\int_{-\infty}^{\infty} e^{-\beta[(a-2d)\phi^2+\lambda\phi^4]}d\phi\right]^{|\Lambda|} < \infty \quad \text{if } \lambda > 0 \text{ or } \ \lambda = 0 \text{ and } a-2d > 0.$$

Call $a - 2d = \frac{r}{2}$ and so

$$\mathcal{H}_\Lambda(\phi) = \frac{1}{2}\sum_{\langle xy \rangle}(\phi_x - \phi_y)^2 + \frac{r}{2}\sum \phi_x^2 + \lambda \sum \phi_x^4. \tag{6.3}$$

<u>Remark</u>. The Ising model is a limit of this model : take

$$\frac{r}{2} = -2\lambda \text{ so } \quad \frac{r}{2}\phi_x^2 + \lambda\phi_x^4 = \lambda(\phi_x^2 - 1)^2 - \lambda$$

and $\sqrt{\frac{\lambda}{\pi}}e^{-\lambda(\phi^2-1)^2} \underset{\lambda\to\infty}{\longrightarrow} \delta(\phi^2-1)$ (for suitable correlations).

7 Gaussian Integrals

7.1 Definitions and elementary properties

Consider first the case $\lambda = 0$. Then our measure is *Gaussian*.

Definition. Let A be a real symmetric $n \times n$ matrix which is (strictly) positive definite (i.e. all eigenvalues > 0 i.e. $(\phi, A\phi) > 0 \ \ \forall \phi \in \mathbf{R}^n, \phi \neq 0$).

<u>The Gaussian measure on $\mathbf{R}^n$ with covariance A^{-1} and mean 0</u> is the probability measure

$$d\mu(\phi) = \frac{1}{Z}e^{-\frac{1}{2}(\phi,A\phi)}D\phi$$

where $\phi = (\phi_1, \cdots, \phi_n)$, $D\phi = \prod_i d\phi_i$, $(\phi, A\phi) = \sum_{ij}\phi_i A_{ij}\phi_j$.

Some calculations:

1. $Z = \int_{\mathbf{R}^n} e^{-\frac{1}{2}(\phi,A\phi)}D\phi$; let $S \in SO(n), S^TAS = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{pmatrix}$ so det $S = 1$. Then, let $\phi = S\Psi, D\phi = D\Psi$. det $S = D\Psi$,

$$\begin{aligned} Z \ &= \int e^{-\frac{1}{2}\sum_i \lambda_i \Psi_i^2} D\Psi = \prod_{i=1}^{n} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\lambda_i\Psi_i^2}d\Psi_i = \prod_{i=1}^{n}(\frac{2\pi}{\lambda_i})^{1/2} \\ &= [\det(A/2\pi)]^{-1/2} \end{aligned}$$

2. Correlations : $\langle \phi_A \rangle \equiv \langle \prod_{i\in A}\phi_i \rangle$ can be computed using the <u>generating function</u>: Let $f \in \mathbf{R}^n$. Define

$$S(f) = \langle e^{(\phi,f)} \rangle = \frac{1}{Z}\int e^{-\frac{1}{2}(\phi,A\phi)+(\phi,f)}D\phi.$$

Then $S(f)$ is smooth in f and

$$\langle \phi_A \rangle = \frac{\partial^{|A|}}{\partial f^A} \mid_{f=0} S(f) \equiv \prod_{i\in A}\frac{\partial}{\partial f_i} \mid_{f=0} S(f)$$

But

$$\begin{aligned} S(f) &= \frac{1}{Z}\int e^{-\frac{1}{2}(\phi-A^{-1}f,A(\phi-A^{-1}f))+\frac{1}{2}(f,A^{-1}f)}D\phi \\ &= e^{\frac{1}{2}(f,A^{-1}f)}\frac{1}{Z}\int e^{-\frac{1}{2}(\Psi,A\Psi)}D\Psi = e^{\frac{1}{2}(f,A^{-1}f)} \end{aligned}$$

writing $\Psi = \phi - A^{-1}f$.
Hence, say the 2-point function,

$$\langle \phi_i\phi_j\rangle = \frac{\partial^2}{\partial f_i \partial f_j}|_0\ e^{\frac{1}{2}(f,A^{-1}f)} = (A^{-1})_{ij}.$$

In general,

$$\begin{aligned} \langle \prod_{\alpha=1}^{n}\phi_{i_\alpha}\rangle &= \begin{cases} 0 & n \text{ odd} \\ (\prod \frac{\partial}{\partial f_{i_\alpha}})|_0[\frac{1}{2}(f,A^{-1}f)]^m\frac{1}{m!} & n=2m \end{cases} \\ &= \sum_P \prod_{\langle\alpha\beta\rangle\in P}(A^{-1})_{i_\alpha i_\beta} \end{aligned}$$

where P is a pairing of the set $\{1,2,\cdots,2m\}$ i.e. a partition into m sets of size 2. The sum runs through all such pairings.

Example $n=4$: $<12><34>, <13><24>, <14><2,3>$ 3 pairings.
In general, $\#$ of pairings is $\frac{(2m)!}{2^m m!}$ (show!).
Thus

$$\langle\prod_{\alpha=1}^{2m}\phi_{i_\alpha}\rangle = \sum_P \prod_{\langle\alpha,\beta\rangle\in P}\langle\phi_{i_\alpha}\phi_{i_\beta}\rangle$$

a sum of products of 2-point functions.

Conversely, let μ be a probability (Borel) measure on $\mathbf{R}^n$, such that $e^{(\phi,f)}\in L^1(\mu)\ \forall f\in\mathbf{R}^n$ (i.e. $\int e^{(\phi,f)}d\mu<\infty\ \forall f$).
Then we call $S(f)=\int e^{(\phi,f)}d\mu(\phi)$ the *generating function* of μ. In particular,

Lemma $S(f)$ is analytic in $\mathbf{C}^n$, $\langle\phi_A\rangle$ exists and is given by $\frac{\partial^{|A|}}{\partial f^A}|_0 S(f)$.

Proof. Exercice

7.2 Bochner's Theorem

When is a function $S(f)$ a generating function of a measure? For this, it is convenient to study, $S(if), f\in\mathbf{R}^n$ which always exists. This too is often called a generating function

(or characteristic function).
Thus, let μ be a probability measure on $\mathbf{R}^n$, and let

$$W(f) = \int e^{i(\phi,f)} d\mu(\phi) \quad f \in \mathbf{R}^n$$

W is clearly well defined,

a) $|W(f)| \leq \int d\mu = 1$ i.e. is bounded.

b) $W(0) = 1$.
Also, by the dominated convergence theorem, if $f_n \to f$ then, since $|e^{i(\phi,f_n)}| \leq 1, W(f_n) \to W(f)$ so,

c) W is continuous.

Finally, let $z_\alpha \in \mathbf{C}$, $f_\alpha \in \mathbf{R}^n$, $\alpha = 1, \cdots, N$. Then,

d)

$$\begin{aligned} \sum_{\alpha,\beta} z_\alpha \bar{z}_\beta W(f_\alpha - f_\beta) &= \int \sum z_\alpha \bar{z}_\beta e^{i(\phi,f_\alpha)} e^{-i(\phi,f_\beta)} \\ &= \int |\sum z_\alpha e^{i(\phi,f_\alpha)}|^2 d\mu \geq 0 \end{aligned}$$

Definition $W : \mathbf{R}^n \to \mathbf{C}$ is a function of *of positive type* if a) - d) hold.

Theorem. $W(f)$ is of positive type if and only if there is a Borel probability measure on $\mathbf{R}^n$ such that $W(f) = \int e^{i(\phi,f)} d\mu(\phi)$.
(Bochner's theorem).

Proof $\Leftarrow$ is done,
"$\Rightarrow$". One uses an idea that is useful in other contexts.

1. Use W to construct a scalar product : Let

$$\mathcal{H} = \{\psi : \mathbf{R}^n \to \mathbf{C} | \psi(x) \neq 0 \text{ for finitely many } x's\}$$

For $\phi, \psi \in \mathcal{H}$ put

$$(\phi, \psi) = \sum_{x,y \in \mathbf{R}^n} \bar{\phi}(x)\psi(y)W(x-y)$$

(this is a finite sum). (,)is an inner product except that $(\phi, \phi) = 0$ does not imply $\phi = 0$. Let $\mathcal{H}_0 = \{\phi | (\phi, \phi) = 0\}$; $\mathcal{H}_0$ is a subspace of $\mathcal{H}$. Put $\tilde{\mathcal{H}} = \mathcal{H}/\mathcal{H}_0$ = equivalence classes $[\phi] = [\phi + \phi_0], \phi_0 \in \mathcal{H}_0$. Then $(\tilde{\mathcal{H}}, (\cdot, \cdot))$ is an inner product space which can be completed into a Hilbert space.

2. Let for $t \in \mathbf{R}^n, U_t : \mathcal{H} \to \mathcal{H} \quad (U_t\psi)(x) = \psi(x+t)$. Clearly, $U_t : \mathcal{H}_0 \to \mathcal{H}_0$ so one defines $\tilde{U}_t : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ by $\tilde{U}_t[\phi] = [U_t\phi]$. $(\tilde{U}_t\phi, \tilde{U}_t\psi) = (\phi, \psi)$ so $\tilde{U}_t$ is an isometry. We have $\tilde{U}_{t+s} = \tilde{U}_t\tilde{U}_s$, $\tilde{U}_0 = \mathbb{1}$. Also $t \to \tilde{U}_t$ is strongly continuous (i.e. $\forall\psi \in \tilde{\mathcal{H}}\ t \to \tilde{U}_t\psi \in \tilde{\mathcal{H}}$ is continuous from $\mathbf{R}^n \to \tilde{\mathcal{H}}$) because W is continuous:

$$
\begin{aligned}
&\| \tilde{U}_t\psi - \tilde{U}_s\psi\|^2 \\
&= 2(\psi,\psi) - (\tilde{U}_t\psi, \tilde{U}_s\psi) - (\tilde{U}_s\psi, \tilde{U}_t\psi) \\
&= [2(\psi,\psi) - (\psi, \tilde{U}_{t-s}\psi) - (\psi, \tilde{U}_{s-t}\psi)] =
\end{aligned}
$$

$\sum_{x,y} \bar{\psi}(x)\psi(y)[2W(x-y) - W(x-y+t-s) - W(x-y+s-t] \to 0$ as $t \to s$.

We have verified the assumptions of

Stone's Theorem. Let $\mathbf{R}^n \ni t \to U_t$ unitary in Hilbert space be strongly continuous and $U_tU_s = U_{t+s}$, $\forall s,t$, $U_0 = \mathbb{1}$. Then for each $\phi \in \mathcal{H}, \|\phi\| = 1\ \exists$ measure (Borel, probability) μ_ϕ on $\mathbf{R}^n$ such that $(\phi, U_t\phi) = \int e^{i(\lambda,t)}d\mu_\phi(\lambda)$.

Proof See Reed-Simon, vol. 1, Th. VIII.12. [The "reason"is that if $\mathcal{H}$ were finite dimensional we would easily prove U_t differentiable in t, $\frac{\partial U_t}{\partial t_j}|_{t=0} = iA_j$, A_j Hermitean, $[A_i, A_j] = 0$ and $U_t = e^{i\sum t_jA_j}$. Diagonalize all A_j simultaneously $\Rightarrow A_j = \begin{pmatrix} \lambda_1^{(j)} & & 0 \\ & \ddots & \\ 0 & & \lambda_n^{(j)} \end{pmatrix} \Rightarrow$

$d\mu_\phi(\vec{\lambda}) = \sum_{i=1}^n (\vec{\lambda}^{(i)} - \vec{\lambda})|\phi_i|^2$.

To finish the proof of the Theorem, using Stone's Theorem, take $\tilde{\phi} \in \tilde{\mathcal{H}}$

$$
\tilde{\phi} = [\phi], \qquad \phi(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}
$$

$$
W(t) = \sum_{x,y \in \mathbf{R}^n} \delta(x)\delta(y+t)W(x-y) = (\tilde{\phi}, \tilde{U}_t\tilde{\phi}) = \int e^{i(\lambda,t)}d\mu_{\tilde{\phi}}(\lambda)
$$

Some terminology:

Let μ be a measure of total mass one on a σ-algebra $\sum$ on some set $\mathcal{M}$. We say that μ is a *probability measure*. $A \in \sum$ are called *events* and $\mu(A) =$ probability of A.
Let $f : \mathcal{M} \to \mathbf{R}$ be measurable. We say that f is a *random variable*. The *distribution* of f is the probability measure ν on $\mathbf{R}$ with $\nu(B) = \mu(f^{-1}(B))$. The *mean* of f is $\int f d\mu \equiv \langle f \rangle$ and the *variance* of f is $\int f^2 d\mu - (\int f d\mu)^2$. Thus, in our Gaussian example, each $\phi_i : \mathbf{R}^n \to \mathbf{R}$ $\phi_i(\phi) = \phi_i$ is a random variable, with mean 0 and variance $\langle \phi_i^2 \rangle = (A^{-1})_{ii}$. Also, (ϕ, f) is a random variable, with variance $(f, A^{-1}f)$.

7.3 The Gaussian Ginzgurg-Landau model

Consider our $\lambda = 0$ measure defined by (6.1, 6.2): μ_Λ is on $\mathbf{R}^{|\Lambda|}$ and is Gaussian with $(\phi, A\phi) = \sum_{\langle xy\rangle}(\phi_x - \phi_y)^2 + r\sum_{x\in\Lambda}\phi_x^2$. The quadratic form $\sum_{\langle xy\rangle}(\phi_x - \phi_y)^2$ defines a matrix Δ_f (lattice-Laplacean with free b.c. on $\partial\Lambda$) :

$$\sum_{\substack{\langle xy\rangle \\ x,\, y\, \in\, \Lambda}} (\phi_x - \phi_y)^2 = -(\phi, \Delta_f\phi).$$

Concretely :

$$2\sum_{\langle xy\rangle}\phi_x\phi_y = \sum_{x\in\Lambda}\ \sum_{\substack{|u|\,=\,1 \\ x\,+\,u\,\in\,\Lambda}} \phi_x\phi_{x+u}$$

so

$$-(\Delta_f\phi)_x = \begin{cases} 2d\phi_x - \displaystyle\sum_{|u|=1} \phi_{x+u} \text{ if } \mathrm{dist}(x,\Lambda^c) > 1 \\ 2d\phi_x - \displaystyle\sum_{\substack{|u|\,=\,1 \\ x\,+\,u\,\in\,\Lambda}} \phi_{x+u} \text{ if } \mathrm{dist}(x,\Lambda^c) = 1 \end{cases}$$

Example in $d = 1$ we have $(-\Delta\phi)_x = 2\phi_x - \phi_{x-1} - \phi_{x+1}$ if $|x| < L, -(\Delta\phi)_L = 2\phi_L - \phi_{L-1}$ etc.

Other b.c. The periodic one is the most convenient: For this, let us consider the infinite volume first. On $\mathbf{R}^{\mathbf{Z}^d}$, define $(\Delta\phi)_x = \sum_{|u|=1}(\phi_{x+u} - \phi_x)$. Then Δ is self adjoint on $\ell^2(\mathbf{Z}^d) \subset \mathbf{R}^{\mathbf{Z}^d}$:

$$\ell^2(\mathbf{Z}^d) = \{\phi : \mathbf{Z}^d \to \mathbf{C} | \sum_{x\in\mathbf{Z}^d} |\phi_x|^2 < \infty\}$$

Scalar product $(\phi, \psi) = \sum_{x\in\mathbf{Z}^d}\bar{\phi}_x\psi_x$.
So $(\phi, \Delta\psi) = (\Delta\psi, \phi)$ and, for ϕ real,

$$(\phi, \Delta\phi) = -\sum_{\langle xy\rangle}(\phi_x - \phi_y)^2.$$

Let $L \in \mathbf{N}$ be odd (for convenience), and consider $F_L^{per} \subset \ell^2(\mathbf{Z}^d)$ periodic functions : [we use ϕ_x or $\phi(x)$ below!]

$$\phi \in F_L^{per} : \phi(x + Ln) = \phi(x) \quad \forall n \in \mathbf{Z}^d$$

Clearly F_L^{per} can be identified with $\mathbf{R}^{\Lambda_L}$, namely ϕ is determined by $\phi|_{\Lambda_L}, \Lambda_L = \{x| \, |x_i| < \frac{L}{2}\}$. Clearly $\Delta : F_L^{per} \to F_L^{per}$ so, call Δ_{per} the corresponding matrix in $\mathbf{R}^{\Lambda_L}$.
So, e.g. in $d = 1$, $(\Delta_{per}\phi)(x) =$ as above, with $\phi_{[\frac{L}{2}]+1} \equiv \phi_{-[\frac{L}{2}]}$ etc.

Diagonalize Δ_{per}, Δ. By Fourier series. Consider $p \in [-\pi, \pi]^d$ $\phi_p \in \mathbf{R}^{\mathbf{Z}^d}$ $\phi_p(x) = e^{i(p,x)}$. Then,

$$\Delta\phi_p(x) = \sum e^{i(p,x+u)} - e^{i(p,x)} = \sum_{|u|=1}(e^{i(p,u)} - 1)e^{i(p,x)}$$

$$= 2\sum_{\mu=1}^{d}(\cos p_\mu - 1)\phi_p(x) \equiv -\mu(p)\phi_p(x).$$

Now note that $\{\phi_p | p_i = \frac{n_i}{N}\pi \quad n_i \in \mathbf{Z}, |n_i| \le N\}$ where $L = 2N+1$, is a *basis* for F_L^{per} (they are independent and there are L^d of them). We have

$$\sum_p e^{ip(x-y)} = L^d \delta_{xy} \quad \text{for} \quad x, y \in \Lambda_L \tag{7.1}$$

where p runs in $\{p | p_i = \frac{n_i}{N}\pi, n_i \in \mathbf{Z}, |n_i| \le N\} = B_L$

$$\sum_{x \in \Lambda_L} e^{i(p-q)x} = L^d \delta_{pq} \qquad p, q \in B_L \tag{7.2}$$

so $\{L^{-d/2}\phi_p\}_{p \in B_L}$ is an orthonormal basis of eigenvectors of $-\Delta_{per}$ and so (7.1), (7.2) $\Rightarrow$

$$(-\Delta_{per} + r)^{-1}_{xy} = L^{-d} \sum_{p \in B_L} e^{ip(x-y)} \frac{1}{\mu(p) + r} \equiv G_L(x-y)$$

$$\mu(p) = 2\sum_{j=1}^{d}(1 - \cos p_i).$$

Thus, the correlations of the $\lambda = 0$ GL-model are given in terms of the (periodic b.c.) *Green's function* G_L. How about $L \to \infty$?

1.

$$\lim_{L \to \infty} G_L(x-y) = \int_{[-\pi,\pi]^d} \frac{d^d p}{(2r)^d} \frac{e^{ip(x-y)}}{\mu(p) + r} \equiv G(x-y) \tag{7.3}$$

2. $-\Delta$ has spectrum $\{\mu(p) | p \in [-\pi, \pi]^d\}$. By Fourier series, we get:

If $f \in \ell^2(\mathbf{Z}^d)$, let $\hat{f}(p) = \sum_{x \in \mathbf{Z}^d} e^{-ipx} f(x)$, then $\hat{f} \in L^2(B), B = [-\pi, \pi]^d$ and

$$(f, g) \equiv \sum_{x \in \mathbf{Z}^d} \overline{f(x)} g(x) = \int_B \frac{d^d p}{(2\pi)^d} \overline{\hat{f}(p)} \hat{g}(p)$$

$$f(x) = \int_B e^{ipx} \hat{f}(p) \frac{d^d p}{(2\pi)^d}$$

Thus,

$$((-\Delta + r)^{-1} f)(x) = \int e^{ipx} \frac{1}{\mu(p) + r} \hat{f}(p) \frac{d^d p}{(2\pi)^d}$$

$$= \sum_y G(x-y) f(y) \quad \text{so} \quad \left(\frac{1}{-\Delta + r}\right)_{xy} = G(x-y)$$

We have obtained

Proposition Let μ_L be the Gaussian measure on $\mathbf{R}^{\Lambda_L}$, with covariance $(-\Delta_{per}+r)^{-1}$. Then

$$\int \prod \phi(x_\alpha) d\mu_L(\phi) \underset{L\to\infty}{\longrightarrow} \sum_P \prod_{\langle\alpha\beta\rangle} G(x_\alpha - x_\beta)$$

Can we talk of the RHS in terms of a measure in $\mathbf{R}^{\mathbf{Z}^d}$ or $\ell^2(\mathbf{Z}^d)$? To answer this question, we have to make a detour through the theory of measures on spaces of distributions.

8 Measures on function spaces

The answer to the question is that, if one can prove sufficiently strong properties of the limit (Gaussian is a simple case since it is explicit) one automatically has a measure by a Bochner-type theorem.
The main problem is the non-compactness of $\mathbf{R}$: before, in spin systems, convergence of correlations automatically gave a measure by Rietz's theorem. Here things go badly wrong: $\mathbf{R}^{\mathbf{N}}$ is not even locally compact.

Suppose we had a measure μ on some σ-algebra $\mathcal{A}$ of subsets of $\mathbf{R}^{\mathbf{N}}$ such that sets of the form $A \times \mathbf{R}^{\mathbf{N}\setminus n} \in \mathcal{A}$ where $A \in \mathcal{A}_n$ = some σ-algebra of subsets of $\mathbf{R}^n$ and $\mathbf{R}^{\mathbf{N}\setminus n}$ denotes $\times_{i=n+1}^{\infty}\mathbf{R}$ (and $\mathbf{R}^{\mathbf{N}} \equiv \times_{i=1}^{\infty}\mathbf{R}$). Then we get a measure μ_n on $\mathcal{A}_n$ by "integrating out"other variables :

$$\mu_n(A) = \mu(A \times \mathbf{R}^{\mathbf{N}\setminus n}) \tag{8.1}$$

This resulting set of measures is *consistent* i.e.

$$\mu_n(A) = \mu_{n+m}(A \times \mathbf{R}^m) \tag{8.2}$$

provided $A \times \mathbf{R}^m \subset \mathcal{A}_{n+m}$ if $A \in \mathcal{A}_n$. Thus, conversely, let $\{\mu_n\}_{n=1}^{\infty}$ be a family of measures, μ_n a Borel probability measure on $\mathbf{R}^n$. We say μ_n *are consistent if* (8.2) holds for all Borel sets $A \subset \mathbf{R}^n$. A natural σ-algebra on $\mathbf{R}^{\mathbf{N}}$ is the σ-algebra generated by the *cylinder sets*. Let $\mathcal{B}$ = Borel sets on $\mathbf{R}$, $\mathcal{B}^n$ = Borel on $\mathbf{R}^n$.

Definition. A *cylinder set* on $\mathbf{R}^{\mathbf{N}}$ is a set of the form $S_{B_1,\cdots,B_n} = \{x \in \mathbf{R}^{\mathbf{N}} | x_i \in B_i, i = 1,\cdots,n, B_i \in \mathcal{B}, n < \infty\}$. The σ-algebra of subsets of $\mathbf{R}^{\mathbf{N}}$ generated by cylinder sets is denoded by $\mathcal{B}^{\mathbf{N}}$.
A measure μ defined on $\mathcal{B}^{\mathbf{N}}$ is called a *cylinder measure.*

<u>Kolmogorov's extension theorem</u> Let $\{\mu_n\}_{n=1}^{\infty}$ be a consistent family of measures on $\mathcal{B}^n$. Then there is a unique cylinder measure μ on $\mathcal{B}^{\mathbf{N}}$ such that

$$\mu(S_{B_1,\cdots,B_n}) = \mu_n(B_1 \times B_2 \times \cdots \times B_n) \tag{8.3}$$

Proof Main problem is countable additivity. We can do this with Rietz theorem : Let $\dot{\mathbf{R}}$ = one -point compactification of $\mathbf{R}$ (i.e. $\dot{\mathbf{R}} = \mathbf{R} \cup \{\infty\}$ and open sets are open sets in $\mathbf{R}$ and sets of the form $A \cup \{\infty\}$ where $A \subset \mathbf{R}$ is open and $A^c \subset [-n, n]$ some $n < \infty$, $\dot{\mathbf{R}}$ is easily seen to be compact).
Let $\mathcal{M} = \times_{i=1}^{\infty} \dot{\mathbf{R}} = \dot{\mathbf{R}}^{\mathbf{N}}$; $\mathcal{M}$ is compact. Let
$C(\mathcal{M}) = \{f : \mathcal{M} \to \mathbf{R} | f \text{ continuous }\}$,
$C_0(\mathcal{M}) = \{f : \mathcal{M} \to \mathbf{R}|,\ f \text{ depends on finitely many } x_i\}$.

Define, for $f \in C_0(\mathcal{M})$,

$$\ell(f) = \int f d\mu_n$$

where n is large enough, and we extend μ_n to $\dot{\mathbf{R}}^n$ by putting $\mu_n(B_1 \times \cdots \times \{\infty\} \times \cdots \times B_n) = 0$. Consistency : ℓ is n independent if n is large enough. So ℓ defines a positive linear functional $\ell : C_0(\mathcal{M}) \to \mathbf{R}$, $\ell(1) = 1$, $|\ell(f)| \le \|f\|_\infty \Rightarrow \ell$ extends to $C(\mathcal{M})(C_0(\mathcal{M})$ is dense) $\Rightarrow_{\text{Rietz}} \exists$ unique Borel-measure μ on $\mathcal{M}$ such that $\ell(f) = \int f d\mu$. Recall that $\mathcal{M} = \dot{\mathbf{R}}^{\mathbf{N}}$. But $\mu(\{x \in X | x_n = \infty \text{ some } n) \le \sum_{n=1}^{\infty} \mu(\{x | x_n = \infty\}) = \sum_{n=1}^{\infty} \mu_n(\{x | x_n = \infty\}) = 0$. Hence μ is supported on $\mathbf{R}^{\mathbf{N}}$. By construction (8.3) holds ($\mathcal{B}^{\mathbf{N}} \subset$ Borel sets of $\dot{\mathbf{R}}^{\mathbf{N}}$)[2].

As it turns out, we can embed all interesting function (and distribution) spaces into $\mathbf{R}^{\mathbf{N}}$. Let us start with two subsets :

8.1 Minlos Theorem : Measures on $\mathbf{s}'$.

Let

$$\mathbf{s}_m = \{x \in \mathbf{R}^{\mathbf{N}} | \sum_{n=1}^{\infty} n^{2m} |x_n|^2 \equiv \|x\|_m^2 < \infty\}$$

for $m \in \mathbf{Z}$. Let

$$\mathbf{s} = \bigcap_{m \in \mathbf{Z}} \mathbf{s}_m \qquad \mathbf{s}' = \bigcup_{m \in \mathbf{Z}} \mathbf{s}_m$$

(note : $\mathbf{s}_m \supset \mathbf{s}_{m+1} \supset \cdots$). $\mathbf{s}$ is the set of sequences which decay as $n \to \infty$ faster than any power, s' the ones with at most polynomial growth. Give $\mathbf{s}$ the topology generated by the neighbourhoods of 0 of the form

$$N(m, \epsilon) = \{x |\ \|x\|_m < \epsilon\}$$

(so neighbourhoods of $y \in \mathbf{s}$ are $y + N(m, \epsilon)$).

Lemma The space $\mathbf{s}$ with the above topology is a complete metrizable space, i.e. $\mathbf{s}$ has a complete metric d such that d gives the same topology as above.

[2] Borel sets of $\dot{\mathbf{R}}^{\mathbf{N}}$ = smallest σ-algebra containing open sets and cylinder sets are generated by open sets.

Proof Define the metric

$$d(x,y) = \sum_{m=1}^{\infty} 2^{-m} \frac{\|x-y\|_m}{1+\|x-y\|_m}.$$

Exercise :

a) Show same topology.

b) Show that $\mathbf{s}$ is complete (Cauchy sequences converge).

[By the way, $\mathbf{s}$ is an example of a *Frechet space*].

Lemma $\mathbf{s}'$ is the dual space of $\mathbf{s}$ i.e. the space of continuous linear functionals on $\mathbf{s}$.

Proof Let $y \in \mathbf{s}'$. y defines a linear functional ℓ_y on s : $\ell_y(x) \equiv (y,x) \equiv \sum_{n=1}^{\infty} y_n x_n$.
This converges since $|y_n| \le Cn^m$ some $m < \infty$ and $|x_n| \le C_m n^m$ for all m.
ℓ_y is continuous: let $d(x^{(k)},x) \xrightarrow[k\to\infty]{} 0$. Then

$$\begin{aligned} |\ell_y(x^{(k)}) - \ell_y(x)| &\le \sum_{n=1}^{\infty} Cn^m |x_n^{(k)} - x_n| = C \sum n^{m+1} |x_n^{(k)} - x_n| \frac{1}{n} \\ &\le C(\sum \frac{1}{n^2})^{1/2} \|x^{(k)} - x\|_{m+1} \\ &\le C' 2^{m+1} d(x^{(k)}, x) \to 0 \text{ as } k \to \infty. \end{aligned}$$

Conversely : let $\ell : \mathbf{s} \to \mathbf{R}$ be continuous, linear. Let $e^{(m)} \in \mathbf{s} : e_n^{(m)} = \delta_{nm}$. Put $y_m = \ell(e^{(m)})$. Given $x \in \mathbf{s}$, let $x^{(n)} = \sum_{m=1}^{n} e^{(m)} x_m$. Then $x^{(n)} \to x$ in $\mathbf{s}$ (prove!) and so $\ell(x^{(n)}) \to \ell(x)$. But $\ell(x^{(n)}) = \sum_{m=1}^{n} y_m x_m$. Since ℓ is continuous, there exists $C < \infty$, $m < \infty$ such that $|\ell(z)| \le C\|z\|_m$ $\forall z$ (Proof : $\exists$ neighbourhood U of zero in $\mathbf{s}$ such that $|\ell(U)| < 1$, but U contains a set $\{z | \; \|z\|_m < \delta\}$ some $m, \delta \Rightarrow$ given $z \in \mathbf{s}, \delta \frac{z}{\|z\|_m} \in U \Rightarrow |\ell(\delta \frac{z}{\|z\|_m})| < 1$ i.e.

$$\frac{\delta}{\|z\|_m} |\ell(z)| < 1 \Rightarrow \ell(z) \le \delta^{-1} \|z\|_m).$$

Hence $|\ell(e^{(n)})| \le C\|e^{(n)}\|_m = Cn^m$ hence $|y_m| \le Cn^m$ hence $y \in \mathbf{s}'$ and $\ell(x) = (y,x)$

The reason we introduced $\mathbf{s}$ and $\mathbf{s}'$ is the following :
Let μ be a cylinder (probability) measure on $\mathcal{B}^{\mathbf{N}}$. Note that $\mathbf{s}'$ is a μ-measurable set (prove !). Suppose $\mu(\mathbf{s}') = 1$ i.e. $\mu(\mathcal{B}^{\mathbf{N}} \backslash \mathbf{s}') = 0$ i.e. μ is supported in $\mathbf{s}'$. Then,

Exercise : Let $x \in \mathbf{s}$. Then the function $y \to (y,x)$ from $\mathbf{s}'$ to $\mathbf{R}$ is μ-measurable.

Definition Let μ be a cylinder probability measure on $\mathbf{s}'$. The characteristic function $W : \mathbf{s} \to \mathbf{C}$ of μ is

$$W(x) = \int_{\mathbf{s}'} e^{i(y,x)} d\mu(y)$$

Lemma W satisfies

a) $W(0) = 1$

b) W is *positive definite*: given $z_i \in \mathbf{C}$, $x^{(i)} \in \mathbf{s}$ $\; i = 1, \cdots, n$ we have

$$\sum_{i,j=1}^{n} \overline{z_i} z_j W(x_i - x_j) \geq 0$$

c) $W : \mathbf{s} \to \mathbf{R}$ is continuous.

Proof As before, for c) use again the dominated convergence theorem

Finally, our goal :

Minlos's theorem A necessary and sufficient condition for a function W on $\mathbf{s}$ to be a characteristic function of a probability measure on $\mathbf{s}$' is that it obeys a)-c) above.

Proof Let W be given. Let $u \in \mathbf{R}^n$, put $x = (u_1, \cdots, u_n, 0, 0, \cdots) \in \mathbf{s}$.
Then $W(x) \equiv \tilde{W}_n(u)$ is a function of positive type in $\mathbf{R}^n \Rightarrow \exists$ probability measure μ_n on $\mathbf{R}^n$ such that

$$\tilde{W}_n(u) = \int_{\mathbf{R}^n} e^{i(y,u)} d\mu_n(u)$$

$\{\mu_n\}_{n=1}^{\infty}$ are consistent since $\widetilde{W}_{n+m}((u_1 \cdots u_n, 0, \cdots, 0)) = \widetilde{W}_n(u)$ and $\widetilde{W}_n$ uniquely determines μ_n (Why?). Hence $\exists \mu$ on $\mathbf{R}^{\mathbf{N}}$ such that

$$W(x) = \int_{\mathbf{R}^{\mathbf{N}}} e^{i(y,x)} d\mu(y) \tag{8.4}$$

for all $x \in \mathbf{s}$ with only finitely many non zero x_i. We need to show : $\mu(\mathbf{s}') = 1$. In that case the integral can be restricted to $\mathbf{s}'$ and (8.4) holds for all $x \in \mathbf{s}$ (because both sides are continuous in x and the set of x with a finite number of nonzero x_i is dense in $\mathbf{s}$).
$\underline{\mu(\mathbf{s}') = 1}$: Recall that $\mathbf{s}' = \bigcup_{m=-\infty}^{\infty} \mathbf{s}_m$ with

$$\mathbf{s}_m = \{y | \sum_{n=1}^{\infty} n^{2m} y_n^2 \equiv \|y\|_m^2 < \infty\}.$$

We show : Given $\epsilon > 0$, there exists an m such that

$$\mu(\mathbf{s}_m) \geq 1 - \epsilon \tag{8.5}$$

Since $\mu(\mathbf{s}') \geq \mu(\mathbf{s}_m)$ we get $\mu(\mathbf{s}') \geq 1 - \epsilon \; \forall \epsilon$ i.e. the claim.
To prove (8.5), we use first the monotone convergence theorem to get a more managable expression :

$$\lim_{\alpha \downarrow 0} \lim_{N \to \infty} \int e^{-\frac{\alpha}{2} \sum_{n=1}^{N} n^{2m} y_n^2} d\mu = \lim_{\alpha \downarrow 0} \int e^{-\frac{\alpha}{2} \|y\|_m^2} d\mu = \mu(\mathbf{s}_m)$$

(since $e^{-\frac{\alpha}{2}\|y\|_m^2} \nearrow \chi(y \in \mathbf{s}_m) \equiv \begin{cases} 1 & y \in \mathbf{s}_m \\ 0 & y \notin \mathbf{s}_m \end{cases}$).
We consider $\int e^{-\frac{\alpha}{2}\sum_{n=1}^{N} n^{2m} y_n^2} d\mu$. Choose m:
Continuity of $W(x) \Rightarrow \exists m, \delta$ such that

$$|W(x) - 1| \leq \epsilon \ \text{ if } \ \|x\|_{-m-1}^2 \leq \delta \tag{8.6}$$

(recall that $W(0) = 1$). (note that m here tends to be < 0 !).
Then, for all $x \in \mathbf{s}$,

$$\text{Re } W(x) \geq 1 - \epsilon - \frac{2}{\delta}\|x\|_{-m-1}^2 \tag{8.7}$$

since if $\|x\|_{-m-1}^2 < \delta$, (8.7) holds by (8.6) and if $\|x\|_{-m-1}^2 > \delta$, (8.7) holds because $|W(x)| \leq W(0) = 1$ implies Re $W(x) \geq -1$.
Now write

$$e^{-\frac{\alpha}{2}\sum_{n=1}^{N} n^{2m} y_n^2} = \int_{\mathbf{R}^N} e^{i(y,x)} d\nu(x)$$

$$d\nu(x) = \prod_{n=1}^{N} (2\pi\alpha n^{2m})^{-1/2} \exp\left(-\frac{x_n^2}{2\alpha n^{2m}}\right) dx_n$$

to get

$$\int e^{-\frac{\alpha}{2}\sum^{N} n^{2m} y_n^2} d\mu = \int_{\mathbf{R}^N} \left(\int e^{i\sum_1^N y_i x_i} d\mu(y)\right) d\nu(x) = \int_{\mathbf{R}^N} W(x) d\nu(x)$$

Hence by (8.7),

$$\int e^{-\frac{\alpha}{2}\sum^{N} n^{2m} y_n^2} d\mu \geq 1 - \epsilon - \frac{2}{\delta}\int_{\mathbf{R}^N} \|x\|_{-m-1}^2 d\nu(x).$$

But

$$\int_{\mathbf{R}^N} \|x\|_{-m-1}^2 d\nu = \sum_{n-1}^{N} n^{-2m-2} \int x_n^2 d\nu(x) = \sum_{n=1}^{N} n^{-2m-2} \alpha n^{2m} \leq \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)\alpha.$$

Thus

$$\lim_{N\to\infty} \int e^{-\frac{\alpha}{2}\sum_{n=1}^{N} n^{2m} y_n^2} d\mu \geq 1 - \epsilon - \frac{c\alpha}{\delta}$$

and our claim follows

Example Gaussian measures. Let C be a matrix C_{ij} $i, j = 1, 2, \cdots$ with $|C_{ij}| \leq a i^m . j^m$ some $a < \infty, m < \infty$ and $(x, Cx) = \sum x_i x_j C_{ij} > 0$ $\forall x \neq 0$ $x \in \mathbf{s}$ (note (x, Cx) makes sense since $|x_i| < b_n i^{-n}$ all n). Then the function

$$W_C(x) = e^{\frac{1}{2}(x, Cx)}$$

is of positive type.

Proof W is continuous since $|(x,Cx)| \le A\|x\|_n^2$ for some n. Hence $W_C(x) = \lim_{k\to\infty} W_C(x^{(k)})$, $x_i^{(k)} = \begin{cases} x_i & i \le k \\ 0 & i > k \end{cases}$. But

$$W_C(x^{(k)}) = e^{\frac{1}{2}\sum_{ij=1}^k x_i x_j C_{ij}} = \int_{\mathbf{R}^k} e^{i\sum_{i=1}^k x_i y_i} d\mu_k(y)$$

where μ_k is the Gaussian measure on $\mathbf{R}^k$ with covariance matrix C_{ij}, $i,j = 1,\cdots,k$. By assumption, this is a positive $k \times k$ matrix. Hence, let $W_C^k(x) = W_C(x^{(k)})$ $x \in \mathbf{s}$ so W_C^k is of positive type and $W_C^k(x) \to W_C(x)$ $\forall x \in \mathbf{s} \to W_C(x)$ is of positive type .

We call the measure μ_C corresponding to W_C the *Gaussian measure in* $\mathbf{s}'$ *with covariance* C.

Example. Order $\mathbf{Z}^d$ arbitarily such that $\mathbf{Z}^d = \{x_k | k = 1,2,\cdots\}$ and $|x_{k+1}| \ge |x_k|$. Then $\phi : \mathbf{Z}^d \to \mathbf{R}$ becomes $\tilde{\phi} : \mathbf{N} \to \mathbf{R}$ $\tilde{\phi}_i = \phi_{x_i}$. Thus, if C_{xy} $x,y \in \mathbf{Z}^d$ is a positive matrix with $|C_{xy}| \le a(|x|+1)^m(|y|+1)^m$ we can define the Gaussian measure with covariance C on $\mathbf{s}'(\mathbf{Z}^d) = \{\phi : \mathbf{Z}^d \to \mathbf{R} | \sum_{x\in\mathbf{Z}^d}(|x|+1)^{2m}|\phi_x|^2 < \infty$ some $m \in \mathbf{Z}\}$. Thus $C = (-\Delta + r)^{-1}$ is an example (it is positive and $|C_{xy}| \le$ constant; this holds : $r \ge 0, d \ge 3$ and $r > 0, d = 2$).

8.2 Measures on Spaces of Distributions

Let

$$S(\mathbf{R}) = \{f : \mathbf{R} \to \mathbf{R} | f \text{ is } C^\infty \text{ and } \sup_{x\in\mathbf{R}}(1+|x|^m)\frac{d^n f}{dx^n}| < \infty \text{ for all } n,m \ge 0\} \tag{8.8}$$

i.e. f and its derivatives decay faster than any power. Let us define ("creation and annihilation operators")

$$a = \frac{1}{\sqrt{2}}(x + \frac{d}{dx}),\ a^+ = \frac{1}{\sqrt{2}}(x - \frac{d}{dx})$$

and $N = a^+a = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 - \frac{1}{2}$. For $f \in S(\mathbf{R})$, let

$$\|f\|_m = (\int_{\mathbf{R}} |(N+1)^m f|^2 dx)^{1/2}$$

(this is finite due to (8.8)[3].
We give S the topology where neighbourhoods of 0 are

$$N(m,\epsilon) = \{f \in S| \ \ \|f\|_m < \epsilon\}$$

Proposition S is homeomorphic to $\mathbf{s}$.

Proof Consider the functions $\phi_n \in S$:

$$\phi_0 = \frac{1}{\pi^{1/4}}e^{-\frac{1}{2}x^2}, \phi_n = \frac{1}{\sqrt{n!}}(a^+)^n\phi_0$$

[3] See Remark 1 below

These satisfy :

$$(\phi_n, \phi_m) \equiv \int_{-\infty}^{\infty} \phi_n \phi_m dx = \delta_{nm}$$

Proof $a\phi_0 = 0$, $aa^+ = a^+a + 1$ imply

$$a\phi_n = \frac{1}{\sqrt{n!}} a(a^+)^n \phi_0 = \frac{1}{\sqrt{n!}}[(a^+)^{n-1}\phi_0 + a^+a(a^+)^{n-1}\phi_0]$$

$$= \cdots = \frac{n}{\sqrt{n!}}(a^+)^{n-1}\phi_0 \Rightarrow a^n\phi_n = \frac{n!}{\sqrt{n!}}\phi_0$$

But also, for $\phi, \psi \in S, (a^+\phi, \psi) = (\phi, a\psi)$ so $(\phi_n, \phi_n) = \frac{1}{\sqrt{n!}}(\phi_0, a^n\phi_n) = (\phi_0, \phi_0) = 1$. $n \neq m$ gives 0 since $a\phi_0 = 0$.

The functions ϕ_n are thus orthonormal in $L^2(\mathbf{R})$. They are a basis (they are Hermite functions). Now our map $S \to \mathbf{s}$ is $f \to x = (x_1, x_2, \cdots)$, with $x_n = (f, \phi_{n-1})$. Note that

$$N\phi_n = a^+a\phi_n = n\phi_n$$

so $(N+1)^m\phi \to x'$ with $x'_n = n^m x_n$.
Thus

$$\|f\|_m^2 = \int |(N+1)^m f|^2 dx = \sum_{k=0}^{\infty} |((N+1)^m f, \phi_k)|^2 = \sum_{n=1}^{\infty} n^{2m}|x_n|^2 = \|x\|_m^2$$

i.e. the map $f \to x$ maps the neighbourhoods to each other. To conclude, we need to show that this map is a bijection.

A) Injection : $\{\phi_n\}$ is a basis of L^2.

B) Onto : Let $x \in \mathbf{s}$. Then we need to show that $\sum_{n=1}^{N} x_n\phi_{n-1}$ converges as $N \to \infty$ to an element f of S and $x_n = (f, \phi_{n-1})$. $\sum^N x_n\varphi_{n-1}$ converges in L^2 to $f \in L^2$. $(N+1)^m \sum^N x_n\varphi_{n-1} = \sum^N x_n n^m \varphi_{n-1}$ converges in $L^2 \Rightarrow f \in C^\infty$. And $\|f\|_m = \|x\|_m$ so $f \in S$.

Definition The space of *tempered distributions* on $\mathbf{R}$ is the set $S'(\mathbf{R})$ of continuous linear functionals on $S(\mathbf{R})$.

Let $\varphi \in S'(\mathbf{R})$. Put $y_n = \varphi(\phi_{n-1})$. Then φ continuous means[4] : $\exists C, m : |\varphi(f)| \leq C\|f\|_m$ $\forall f \in S \Rightarrow |y_n| = |\varphi(\phi_{n-1})| \leq C\|\phi_{n-1}\|_m = Cn^m$. Hence $y \in \mathbf{s}'$. Conversely, $y \in \mathbf{s}'$ gives rise to $\varphi_y \in S'(\mathbf{R})$ by $\varphi_y(f) = \sum_{n=1}^{\infty} y_n(f, \phi_n)$. Indeed, from above, $x_n = (f, \varphi_n)$ is in $\mathbf{s}$ so $|\varphi_y(f)| \leq C\|x\|_m$ for some m and this $= C\|f\|_m$.

[4] φ continuous : $\forall\epsilon\ \exists N(m, \delta) : f \in N(m, \delta) \Rightarrow |\varphi(f)| < \epsilon$ i.e. $\|f\|_m < \delta \Rightarrow |\varphi(f)| < \epsilon$. Thus for any $f \in S$, $|\varphi(f)| = |\varphi\left(\frac{f}{\|f\|_m}\delta\right)|\frac{\|f\|_m}{\delta} \leq \frac{\epsilon}{\delta}\|f\|_m = C\|f\|_m$.

Remark 1. We see from above that $\| \ \|_m$ is a *norm* on $S : \|f\|_m^2 = 0 \Rightarrow (f, \varphi_n) = 0 \ \forall n \Rightarrow f = 0$. Moreover, since $\|f\|_m^2 = \Big((a^+a+1)^m f, (a^+a+1)^m f\Big) = \Big(f, (a^+a+1)^{2m} f\Big) = \sum$ of $\Big(f, (a^+a)^k f\Big)$ and $(a^+a)^k f =$ (Polynomial in x and $\frac{d}{dx}$) f, we get

$$\|f\|_m \leq C_m \sum_{\alpha,\beta \leq 2m} \sup_x |x^\alpha \frac{d^\beta}{dx^\beta} f(x)| \equiv C_m \|f\|^{(m)}$$

and conversely (show !). So, $\varphi \in S' \iff \exists C, N : |\varphi(f)| \leq C\|f\|^{(N)}$. This is in practice useful, see Examples 1,2 below.

Example 1 Let φ be a polynomially bounded function : $|\varphi(x)| \leq C(1+|x|)^m$. Then it defines a distribution $\varphi \in S'$ by

$$\varphi(f) = \int \varphi(x) f(x) dx$$

since

$$\begin{aligned} &|\int \varphi(x) f(x) dx| \leq C \int (1+|x|)^m |f(x)| dx \leq \\ &C \int (1+|x|)^{m+2} |f(x)| \frac{1}{[1+|x|]^2} dx \leq C' \sup_x (1+|x|)^{m+2} |f(x)| \leq C'' \|f\|^{(m+2)} \end{aligned}$$

so $\varphi \in S'$.

Example 2 φ can be a "delta function"or its derivatives : let

$$\varphi(f) = \frac{d^k f}{dx^k}(x_0).$$

Then,

$$|\varphi(f)| \leq \|D^k f\|_\infty$$

so $\varphi \in S'$. We denote $\varphi = \frac{d^k}{dx^k} \delta(x = x_0)$.

Remark This works in $\mathbf{R}^n$ too :

$$S(\mathbf{R}^n) = \{f \in C^\infty(\mathbf{R}^n) | \ \|x^\alpha D^\beta f\|_\infty \leq C_{\alpha\beta} \ \ \forall \alpha, \beta\}$$

where $\alpha = (\alpha_1, \cdots, \alpha_n), \beta = (\beta_1, \cdots, \beta_n), x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}, D^\beta = \prod_{i=1}^n \frac{\partial^{\beta_i}}{\partial x_i^{\beta_i}}$. Basis $\{\varphi_k\}_{k \in \mathbf{N}^n}$ $\varphi_k = \prod_{i=1}^n \varphi_{k_i - 1}(x_i)$ and we define $\mathbf{s}^{(n)} = \{x \in \mathbf{R}^{\mathbf{N}^n} | x = (x_k)_{k \in \mathbf{N}^n} \sum_k (\prod k_i)^{2m} |x_k|^2 < \infty$ $\forall m\}$. Do the rest!

Definition *1. The cylinder set σ-algebra in* $S'(\mathbf{R})$ is the smallest σ-algebra containing all sets

$$A(\varphi, B) = \{\varphi \in S'(\mathbf{R}) | \varphi(f) \in B\}$$

for $f \in S, B$ Borel in $\mathbf{R}$. It is just the image of our old cylinder algebra on $\mathbf{s}'$ to S' by the map constructed above. Indeed, the cylinder σ-algebra C in $\mathbf{s}'$ is the smallest σ-algebra of subsets of $\mathbf{s}'$ that contains the sets $\sigma_n(B) = \{y \in \mathbf{s}' | y_n \in B\}$ where B is Borel in $\mathbf{R}$. I.e. it is the smallest σ-algebra such that functions

$$p_n : \mathbf{s}' \to \mathbf{R} \quad , \quad p_n(y) = y_n$$

are measurable (i.e. $p_n^{-1}(B) \in C \ \forall B$ Borel i.e. $\sigma_n(B) \in C \ \forall B$). Let $\tilde{C}$ be the smallest σ-algebra in $\mathbf{s}'$ such that the functions $\pi_x : \mathbf{s}' \to \mathbf{R} \quad \pi_x(y) = (y, x)$ are measurable for all $x \in \mathbf{s}$. Since $p_n = \pi_{e_n^{(n)}}$, $e_m^{(n)} = \delta_{nm}$ we see that $C \subset \tilde{C}$. But, also, $\pi_x = \sum_{n=1}^{\infty} x_n p_n$ so π_x is C-measurable. Hence $\tilde{C} \subset C$. Thus $C = \tilde{C}$. Clearly the σ-algebra defined above in $S'(\mathbf{R})$ is the image of $\tilde{C}$ under the map $y \in \mathbf{s}' \to \varphi_y \in S'$. This proves the claim.

Definition *2. A cylinder measure on* $S'(\mathbf{R})$ is a measure on this σ-algebra. Translating

from $\mathbf{s}, \mathbf{s}'$ to S, S' we get :

Theorem (Minlos) A necessary and sufficient condition for a function W on $S(\mathbf{R}^n)$ to be the generating function of a cylinder probability measure μ on $S'(\mathbf{R}^n)$,

$$W(f) = \int_{S'} e^{i\varphi(f)} d\mu(\varphi)$$

is that $W(0) = 1$, W be of positive type and continuous.

<u>Example</u> Gaussian measures. Here $W(f) = \exp\left[-\frac{1}{2}C(f, f)\right]$ where $C(f, g)$ is a continuous bilinear function on $S(\mathbf{R}^n) \times S(\mathbf{R}^n)$. Important examples are given by integral kernels

$$C(f, g) = \int d^n x \int d^n y C(x, y) f(x) g(y)$$

here $\int |C(x, y)| [(1 + |x|)(1 + |y|)]^m dx dy < \infty$ for some m suffices. We are interested in *translation-invariant* C's i.e. $C(x, y) = G(x - y)$. These are most conveniently expressed in terms of Fourier transform : If $f \in S(\mathbf{R}^n)$ we put

$$\hat{f}(k) = \int f(x) e^{-ikx} d^n x$$

and one gets $\hat{f} \in S(\mathbf{R}^n)$. [This is because $(x^\alpha D^\beta f)^\wedge(k) = i^{\alpha+\beta} D^\alpha k^\beta \hat{f}(k)$, do the details !] Inverse :

$$f(x) = \int e^{ikx} \hat{f}(k) \frac{d^n k}{(2\pi)^n}$$

Then we have

$$\begin{aligned} \int |f(x)|^2 d^n x &= \int |\hat{f}(k)|^2 \frac{d^n k}{(2\pi)^n} \\ (f * g)(x) &\equiv \int f(x - y) g(y) d^n y \\ (f * g)^\wedge(k) &= \hat{f}(k) \hat{g}(k) \end{aligned}$$

and

$$\int f(x) G(x - y) f(y) d^n x d^n y = \int |\hat{f}(k)|^2 \hat{G}(k) \frac{d^n k}{(2\pi)^n}$$

which holds for $f, G \in S$ to start with, but extends to much more general G's.

Example The *free Euclidean field with mass m* is the Gaussian measure μ_G where

$$\widehat{G}(k) = \frac{1}{k^2 + m^2} \quad (k \in \mathbf{R}^n)$$

Clearly $S \ni f \to \int |\hat{f}(k)|^2 \frac{d^n k}{(k^2+m^2)(2\pi)^n}$ is continuous, so

$$W(f) = \exp[-\frac{1}{2} \int \frac{|\hat{f}(k)|^2}{k^2 + m^2} \frac{d^n k}{(2\pi)^n}]$$

determines a measure on $S'(\mathbf{R}^n)$, by Minlos' theorem. Strictly speaking, $G(x)$ is defined as a distribution[5], by the Fourier transform $\widehat{G}$ but actually is a locally integrable function :

$$|G(x)| \le C \frac{e^{-m|x|}}{|x|^{n-2}} \tag{8.9}$$

To see this, let us consider a *ultraviolet cutoff* : let $\Lambda > 0$ and define

$$\widehat{G}_\Lambda(k) = \frac{1}{k^2 + m^2} \frac{\Lambda^2}{k^2 + \Lambda^2}$$

This is called Pauli-Villars cutoff. Note that $\widehat{G}_\Lambda(k) \underset{\Lambda \to \infty}{\longrightarrow} \widehat{G}(k)$ pointwise, and as elements of S'.

Note S' is given a topology such that $\varphi_i \to \varphi$ as $i \to \infty$ if $\varphi_i(f) \to \varphi(f)$ $\forall f \in S$.

For $n < 4$, $\widehat{G}_\Lambda \in L^1(\mathbf{R}^n)$ and so $G_\Lambda(x)$ is a continuous function (for $n \ge 4$: replace $\frac{\Lambda^2}{p^2+\Lambda^2}$ by $\left(\frac{\Lambda^2}{p^2+\Lambda^2}\right)^k$).

So, by rotational symmetry,

$$G_\Lambda(x) = \int \frac{d^n k}{(2\pi)^2} \frac{e^{ik_1|x|}}{k^2 + m^2} \frac{\Lambda^2}{k^2 + \Lambda^2}$$

So the k_1 integral : $k^2 + m^2 = (k_1 + i\sqrt{\vec{k}^2 + m^2})(k_1 - i\sqrt{\vec{k}^2 + m^2}), k^2 + \Lambda^2 = (k_1 + i\sqrt{\vec{k}^2 + \Lambda^2})(k_1 - i\sqrt{\vec{k}^2 + \Lambda^2}), \vec{k} = (k_2, k_3, \cdots)$. By Cauchy:

$$G_\Lambda(x) = \int \frac{d^{n-1}\vec{k}}{(2\pi)^{n-1}} \left[\frac{e^{-\sqrt{\vec{k}^2+m^2}r}}{2\sqrt{\vec{k}^2 + m^2}} \frac{\Lambda^2}{m^2 + \Lambda^2} + \frac{e^{-\sqrt{\vec{k}^2+\Lambda^2}r}}{2\sqrt{\vec{k}^2 + \Lambda^2}} \frac{1}{m^2 + \Lambda^2} \right]$$

which for $r = |x| \neq 0$ is *smooth* and converges as $\Lambda \to \infty$ to

$$G(x) = \int \frac{d^{n-1}\vec{k}}{(2\pi)^{n-1}} \frac{e^{-\sqrt{\vec{k}^2+m^2}}r}{2\sqrt{\vec{k}^2 + m^2}};$$

[5] Fourier transform $\hat{\varphi}$ of a distribution $\varphi \in S'$ is a distribution, $\hat{\varphi} \in S'$, defined by

$$\hat{\varphi}(f) = (2\pi)^n \varphi(\tilde{f}) \quad \tilde{f}(x) = f(-x)$$

which is the usual (see example 1 above) if $\varphi \in S : \int \hat{\varphi}(k) \hat{f}(k) \frac{d^n k}{(2\pi)^n} = \int \varphi(x) f(-x) d^n x$

this is a L^1 function satisfying (8.9).

Note that, a priori, for $\varphi \in S', \varphi(x)$ is not defined, only $\varphi(f),\ f \in S$. These are measurable functions in S' and actually Gaussian random variables on the probability space $(S'(\mathbf{R}^n), \mu)$:

$$\langle e^{it(\varphi,f)} \rangle = e^{-\frac{1}{2}t^2(f,Gf)}$$

so the variance of (φ, f) is (f, Gf). Let $f_i \in S, i = 1, \cdots, N$. Then $x_i \equiv \varphi(f_i)$ are jointly Gaussian :

$$\langle e^{i\sum t_i x_i} \rangle = \langle e^{i\varphi(\sum t_i f_i)} \rangle = e^{-\frac{1}{2}\sum_{ij} t_i t_j (f_i, G f_j)}$$

i.e. $x = (x_1, \cdots, x_N) \in \mathbf{R}^N$ is Gaussian with covariance matrix $A^{-1}_{ij} = (f_i, Gf_j)$. Hence e.g.

$$\int \varphi(f)\varphi(g) d\mu_G(\varphi) = -\frac{d}{dt} |_0 \frac{d}{ds} |_0 \langle e^{-i[t\varphi(f)+s\varphi(g)]} \rangle = (f, Gg) = \int d^n x d^n y f(x) G(x-y) g(y)$$

and, more generally,

$$\int \prod_{i=1}^{2N} \varphi(f_i) d\mu_G(\varphi) = \sum_P \prod_{\langle ij \rangle \in P} (f_i, Gf_j)$$
$$= \int d^n x_1 \cdots d^n x_{2N} f_1(x_1) \cdots f_N(x_N) \sum_P \prod_{\langle ij \rangle \in P} G(x_i - x_j)$$

Hence we may use the notation

$$\int \varphi(x_1) \cdots \varphi(x_{2N}) d\mu_G(\varphi) = \sum_P \prod_{\langle ij \rangle \in P} G(x_i - x_j)$$

even if strictly speaking $\varphi(x)$ is not a well defined random variable.

Remark Indeed, let us try to see what $\varphi(x)^2$ would be. Formally

$$\int \varphi(x)^2 d\mu_G(\varphi) = \lim_{x \to y} \int \varphi(x)\varphi(y) d\mu_G = \lim_{x \to y} G(x-y) = G(0) = \infty.$$

This is a first instance of an *ultraviolet divergence* in quantum field theory. Our G is such that $\varphi(x)$ is not a nice random variable (it is a "distribution valued random variable"). We'll return to this.

9 The Gaussian Ginzburg-Landau Model : Critical Point and Continuum Limit

Let us return to the lattice and the Ginzburg-Landau-model. The $\lambda = 0$ case is the Gaussian measure μ_C on $\mathbf{R}^{\mathbf{Z}^d}$ (or $\mathbf{s}'(\mathbf{Z}^d)$) with covariance

$$C(x-y) = \int \phi(x)\phi(y) d\mu_C(\phi) = \int_{[-\pi,\pi]^d} \frac{e^{ip(x-y)}}{\mu(p) + r} \frac{d^d p}{(2\pi)^d}$$

where $x, y \in \mathbf{Z}^d, \phi \in \mathbf{R}^{\mathbf{Z}^d}$.
We have

$$0 \le C(x-y) \le Ce^{-|x-y|/\xi} \qquad \xi > 0 \quad , \quad \xi \to 0 \text{ as } r \to 0.$$

Proof Let $R = \max_{i=1,\cdots,d} |x_i - y_i|$. We may assume $R = |x_1 - y_1|$.

$$C(x-y) = \int \left(\int_{-\pi}^{\pi} e^{ip_1 R} \frac{1}{2(1-\cos p_1) + \mu(\vec{p}) + r} \frac{dp_1}{2\pi} \right) e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}}$$

Use Cauchy for the p_1 integral PICTURE
A and C cancel since $\cos p_1, e^{ip_1 R}$ are periodic

$$= e^{-\epsilon R} \int \left(\int_{-\pi}^{\pi} e^{ip_1 R} \frac{1}{2(1-\cos(p_1 + i\epsilon)) + \mu(\vec{p}) + r} \right) e^{i\vec{p}(\vec{x}-\vec{y})}$$

Choose $\epsilon : \mathrm{Re} 2(1-\cos(p_1+i\epsilon)) + r \ge 0$ i.e. $2(1-\cos p_1 \cosh \epsilon) + r \ge 0$ i.e. $2\cosh \epsilon \le 2 + r$ or $\epsilon < \cosh^{-1}(1 + r/2)$. [So, for r small, one can choose $\epsilon \sim \sqrt{r}$]. So,

$$C(x-y) \le Ce^{-\epsilon R}$$

Thus $r > 0$ is non critical. For $r = 0$, we have

$$C(x) = |x|^{2-d} \int_{[-\pi|x|, \pi|x|]^d} \frac{e^{ip\frac{x}{|x|}}}{x^2 \mu(p/|x|)} \frac{d^d p}{(2\pi)^d} \equiv |x|^{2-d} h(\hat{x}, |x|)$$

$\hat{x} = x/|x|$,
provided $\boxed{d > 2}$.

<u>Homework</u> Prove that $h(\hat{x}, |x|) \xrightarrow[|x|\to\infty]{} \int \frac{e^{ip_1}}{p^2} \frac{d^d p}{(2\pi)^d}$

$$= \int \frac{e^{-\sqrt{\vec{p}^2}}}{2\sqrt{\vec{p}^2}} \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}} = \frac{A_{d-1}}{2(2\pi)^{d-1}} \int_0^\infty e^{-r} r^{d-3} dr = \frac{\Gamma(d-2) A_{d-1}}{2(2\pi)^{d-1}} \equiv C_d$$

when A_{d-1} = volume of S^{d-2}.

We can state the previous calculation in two ways:

a) We showed $C(x) \sim G(x)$ as $|x| \to \infty$. Here $G(x)$ = 2-point function of the massless free field, $\widehat{G}(p) = \frac{1}{p^2}$. Note that $\widehat{G}$, for $d > 2$, defines a continuous function $e^{-\frac{1}{2}(f, Gf)}$ on S : We have

$$(f, Gg) = \int d^d x d^d y \frac{C_d}{|x-y|^{d-2}} f(x) g(y) \le C \|f\|_m \|g\|_m \quad \text{for} \quad m \quad \text{large enough.}$$

For $d = 2$, see below.

b) Consider the function

$$\begin{aligned}
&L^{d-2}C(Lx-Ly) = L^{d-2}\langle\phi(Lx)\phi(Ly)\rangle \\
&= L^{d-2}\int_{[-\pi,\pi]^d} \frac{e^{ip(Lx-Ly)}}{\mu(p)} d^dp = \int_{[-L\pi,L\pi]^d} \frac{e^{ip(x-y)}}{L^2\mu(p/L)} d^dp \\
&\xrightarrow[L\to\infty]{} \int_{\mathbf{R}^d} \frac{e^{ip(x-y)}}{p^2}\frac{d^dp}{(2\pi)^d} = G(x-y)
\end{aligned}$$

We call this the *scaling limit* : take the statistical mechanics at the critical point, look at long distances, scale ϕ (by $L^{\frac{d-2}{2}}$ above) $\Rightarrow$ get quantum field theory. In this Gaussian model we get : let $x_i \neq x_j$ $i \neq j$, and $\varphi_L(x) = L^{\frac{d-2}{2}}\phi(Lx)$ (for $x \in (L^{-1}\mathbf{Z})^d$).
Then

$$\int_{s'} \prod_{i=1}^{2N} \varphi_L(x_i) d\mu_C(\phi) \xrightarrow[L\to\infty]{} \int_{S'} \prod_{i=1}^{2N} \varphi(x_i) d\mu_G(\varphi)$$

Here the LHS is with a measure defined on variables $\phi(x), x \in \mathbf{Z}^d$, a lattice model and the RHS with a measure defined on variables $\varphi(x)$ $x \in \mathbf{R}^d$, a continuum model (to make this precise, observe that $(L^{-1}\mathbf{Z})^d)$ becomes "dense"in $\mathbf{R}^d$ as $L \to \infty$).

<u>The massive case</u>

How about $r > 0$? Consider again the Ginzburg-Landau model, $C_r = (-\Delta + r)^{-1}$ on $\ell^2(\mathbf{Z}^d)$. So,

$$C_r(x) = |x|^{2-d}\int_{[-|x|\pi,|x|\pi]^d} \frac{e^{ip\frac{x}{|x|}}}{|x|^2\mu(p/|x|) + |x|^2 r}\frac{d^dp}{(2\pi)^d}$$

For $|x| < r^{-1/2}$ (let r be small now)

$$C_r(x) \sim |x|^{2-d}$$

and for $|x| > r^{-1/2}$ scale differently, $p \to r^{1/2}p$,

$$\begin{aligned}
C_r(x) &= r^{\frac{d-2}{2}}\int_{[-\frac{\pi}{r},\frac{\pi}{r}]^d} \frac{e^{ipr^{1/2}x}}{r^{-1}\mu(r^{1/2}p) + 1}\frac{d^dp}{(2\pi)^d} \\
&\sim r^{\frac{d-2}{2}} e^{-r^{1/2}|x|}
\end{aligned}$$

Stated differently, let us put $r = L^{-2}m^2$ and consider

$$\begin{aligned}
L^{d-2}C_{m^2/L^2}(Lx) &= \int_{[-L\pi,L\pi]^d} \frac{e^{ipx}}{L^2\mu(p/L) + m^2}\frac{d^dp}{(2\pi)^d} \\
&\xrightarrow[L\to\infty]{} \int_{\mathbf{R}^d} \frac{e^{ipx}}{p^2 + m^2}\frac{d^dp}{(2\pi)^d} \equiv G_{m^2}(x).
\end{aligned}$$

Hence

$$\int_{\mathbf{s}'} \prod_{i=1}^{2N} L^{\frac{d-2}{2}}\phi(Lx_i) d\mu_{C_{m^2/L^2}}(\phi) \to \int_{S'} \prod \varphi(x_i) d\mu_{G_{m^2}}(\varphi)$$

i.e. if we approach the critical point $r \to 0$ in a suitable way and scale distances and ϕ we get the continuum theory at nonzero m i.e. correlation length $< \infty$.

<u>Remark</u> All the scalings are completely natural :
Formally

$$d\mu_{C_r}(\phi) = \frac{1}{Z} e^{-\frac{1}{2}(\phi,(-\Delta+r)\phi)} \prod_{x\in\mathbf{Z}^d} d\phi(x)$$

and $\phi(x) = L^{\frac{2-d}{2}} \varphi_L(x/L)$ gives

$$\begin{aligned}
(\phi,(-\Delta+r)\phi)) &= \sum_{x\in\mathbf{Z}^d} \phi(x)((-\Delta+r)\phi)(x)) \\
&= \sum_{x\in\mathbf{Z}^d} \phi(x) \left[\sum_{|u|=1} (\phi(x) - \phi(x+u) + r\phi(x) \right] \\
&= \sum_{x\in\mathbf{Z}^d} L^{2-d} \varphi_L(\frac{x}{L}) \left[\sum_{|u|=1} (\varphi_L(\frac{x}{L}) - \varphi_L(\frac{x}{L}+\frac{u}{L})) + r\varphi_L(\frac{x}{L}) \right] \\
&= \sum_{y\in(\frac{1}{L}\mathbf{Z})^d} L^{-d} \varphi_L(y) \left[\sum_{|u|=1} \frac{\varphi_L(y)-\varphi_L(y+\frac{u}{L})}{L^{-2}} + L^2 r \varphi_L(y) \right]
\end{aligned}$$

which, if φ_L went to a smooth function as $L \to \infty$,

$$\underset{\substack{L\to\infty \\ r=m^2/L^2}}{\longrightarrow} \int_{\mathbf{R}^d} \varphi(y)(-\Delta+m^2)\varphi(y) d^d y = (\varphi, G_{m^2}^{-1}\varphi),$$

10 Non-Gaussian Theory : Lattice

10.1 Measures

Let us now consider the $\lambda \neq 0$ Ginzburg-Landau model.
There are various finite volume theories that we can consider : Let $\Lambda \subset \mathbf{Z}^d$ be an L-cube.

A) The measures

$$\frac{1}{\widetilde{Z}_\Lambda} e^{-\lambda \sum_{x\in\Lambda} \phi(x)^4} d\mu_{C_\Lambda}(\phi)$$

on $\mathbf{R}^{|\Lambda|}$ where μ_{C_Λ} is Gaussian with covariance $C_\Lambda = (-\Delta_\Lambda + r)^{-1}$ and Δ_Λ is the Laplacean in $\mathbf{Z}^d$ with some boundary conditions on $\partial\Lambda$ (we have considered periodic and free above; one can consider Dirichlet or Neumann on the lattice too). $\widetilde{Z}_\Lambda$ normalizes the total measure to 1.

B) The measure

$$\frac{1}{Z} e^{-\lambda \sum_{x\in\Lambda} \phi(x)^4} d\mu_C(\phi)$$

on $\mathbf{s}'(\mathbf{Z}^d)$ when μ_C is Gaussian with covariance $C = (-\Delta + r)^{-1}$. Thus μ_C is in infinite volume and ϕ^4 is in finite volume. Note that, whereas in A, $\tilde{Z}_\Lambda = \int e^{-\lambda \sum_{x\in\Lambda} \phi_x^4} d\mu_{C_\Lambda}$ is strictly > 0, in B) this is not immediately obvious here. However, recall Jensen's inequality :

If ν is a probability measure in $\mathbf{R}$, and $f : \mathbf{R} \to \mathbf{R}$ is convex $(f(tx + (1-t)y)) \leq tf(x) + (1-t)f(y)$, $t \in (0,1))$, then

$$\int f(x) d\nu(x) \geq f(\int x d\nu(x))$$

Apply it to the random variable $x = \lambda \sum_{y\in\Lambda} \phi_y^4$ to get

$$\int e^{-\lambda \sum \phi_y^4} d\mu_C \geq \exp[-\lambda \sum_{y\in\Lambda} \langle \phi_y^4 \rangle]$$

where

$$\langle \phi_y^4 \rangle \equiv \int d\mu_C(\phi) \phi_y^4 = 3C(0)^2 = 3 \left[\int \frac{1}{\mu(p) + m^2} \frac{d^d p}{(2\pi)^d} \right]^2$$

from our rules for Gaussian integrals. Also, since $\phi_y^4 > 0$, we get

$$\exp[-3\lambda |\Lambda| C(0)^2] \leq Z_\Lambda \leq 1 \tag{10.1}$$

Note how the extensivity of $Z (= e^{\mathcal{O}(\Lambda)})$ is visible here. Also, note that the free energy: $-3\lambda C(0)^2 \leq \frac{1}{|\Lambda|} \log Z_\Lambda \leq 0$, uniformly in Λ. Similar inequalities hold in the case A) above.

10.2 Perturbation Theory

It is possible to prove quite generally, using Ising-model approximations and various correlation inequalities that the $\Lambda \to \mathbf{Z}^d$ limit of the above measures exist. These are, however, special tricks and we want to understand these issues eventually using the renormalization group. But let us for now proceed more heuristically and study the correlation functions perturbatively in λ. Consider e.g. the pair correlation

$$\langle \phi(x)\phi(y) \rangle_\Lambda \equiv \frac{\int \phi(x)\phi(y) e^{-\lambda \sum_{x\in\Lambda} \phi(x)^4} d\mu_C(\phi)}{\int e^{-\lambda \sum_{x\in\Lambda} \phi(x)^4} d\mu_C(\phi)} \equiv G_2(x,y) \tag{10.2}$$

It is not hard to see that, for $|\Lambda| < \infty$, this is C^∞ in λ : $\frac{d^n}{d\lambda^n} e^{-\lambda \sum \phi^4} = (-\sum \phi^4)^n \; e^{-\lambda \sum \phi^4}$ and

$$(\sum \phi^4)^n e^{-\lambda \sum \phi^4} = n! \epsilon^{-n} \frac{(\epsilon \sum \phi^4)^n}{n!} e^{-\lambda \sum \phi^4} \leq n! \epsilon^{-n} e^{-(\lambda - \epsilon) \sum \phi^4}$$

Using this one shows that both numerator and denominator are C^∞ and from (10.1) above, $Z_\Lambda > 0$. Thus, we can calculate the Taylor coefficients of G_2:

$$G_2(x,y) = \sum_{n=0}^{N} G_{2,n}(x,y) \lambda^n + R_N$$

where $\lim_{\lambda\to 0}\lambda^{-N}R_N(\lambda)=0$. The coefficients $G_{2,n}$ have a nice graphical representation which we now derive.
Let us start with small n. Denote (10.2) by $\frac{\mathcal{N}}{\mathcal{D}}$.

$\underline{n=0}$ $\mathcal{N}=\int\phi(x)\phi(y)d\mu_C+\mathcal{O}(\lambda)$, $\mathcal{D}=1+\mathcal{O}(\lambda)$ so $G_{2,0}=C(x-y)$ our Gaussian covariance.

$\underline{n=1}$ $\mathcal{N}=C(x-y)-\lambda\sum_{z\in\Lambda}\int\phi(x)\phi(y)\phi(z)^4d\mu_C$
$\mathcal{D}=1-\lambda\sum_{z\in\Lambda}\int\phi(z)^4d\mu_C.$
>From our rules of Gaussian integrals:

$$\begin{aligned}\int\phi(z)^4d\mu_C&=C(z-z)C(z-z)+2\text{ others }=3C(0)^2\\ \int\phi(x)\phi(y)\phi(z)^4d\mu_C&=\frac{6!}{2^3.3!}\text{ pairings}=15\text{ terms}\\ &=C(x-y).3C(0)^2+12C(x-z)C(y-z)C(z-z)\end{aligned}$$

So, $\mathcal{N}=C(x-y)(1-3\lambda|\Lambda|C(0)^2)-12\lambda\sum_{z\in\Lambda}C(x-z)C(y-z)C(0)$
$\mathcal{D}=1-3\lambda|\Lambda|C(0)^2$
Hence $G_{2,1}=-12\lambda C(0)\sum_{z\in\Lambda}C(x-z)C(z-y)$.

To proceed further, we need to introduce some notation. Clearly, we need to calculate expressions like

$$\int\phi(x)\phi(y)\prod_{i=1}^N\phi(z_i)^4d\mu_C\text{ and }\int\prod\phi(z_i)^4d\mu_C$$

Both are given as sums of products of pairings : $\sum_P\prod_{\langle\alpha\beta\rangle}C(u_\alpha-u_\beta)$ where $\{u_\alpha\}$ are the points x,y,z_i, however such that each $\underline{z_i\text{ occurs 4 times}}$. To each pairing P we associate a *graph* $\mathcal{G}(P)$ as follows : $\mathcal{G}=(V,\mathcal{L})$ consists of a set V of vertices and a set $\mathcal{L}$ of lines. Vertices are the set $\{x,y,z_1,\cdots,z_N\}=V$ (for the 1st case). Lines ℓ join vertices : $\ell=\{u_{\ell_1},u_{\ell_2}\}$ where $u_{\ell_1},u_{\ell_2}\in V$. $u_{\ell_1}=u_{\ell_2}$ is allowed.
Let $\Gamma_2(V)$ = set of graphs $\mathcal{G}$ such that:

> each vertex z_i has 4 lines attached (where we count lines $\{z_i,z_i\}$ twice) each vertex x,y has 1 line attached

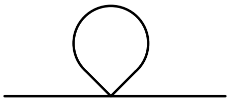
Given a $\mathcal{G}$ satisfying these conditions, there may be several P's such that $\mathcal{G}(P)=\mathcal{G}$. Call the number $\#\{P|\mathcal{G}(P)=\mathcal{G}\}=n(\mathcal{G})$.
Then

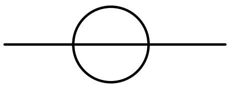
$$\int\prod_{i=1}^N\phi(z_i)^4\phi(x)\phi(y)d\mu_C=\sum_{\mathcal{G}\in\Gamma_2(V)}n(\mathcal{G})\prod_{\ell\in\mathcal{G}}C(u_{\ell_1}-u_{\ell_2})\tag{10.3}$$

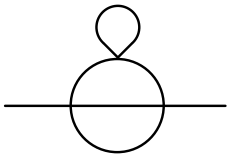
Example In $\langle \phi(z)^4 \phi(x)\phi(y)\rangle$ we had 2 graphs, $\mathcal{G}_1$ has lines $\{z,z\},\{z,z\},\{x,y\}$ and $\mathcal{G}_2$ lines $\{x,z\},\{y,z\},\{z,z\}$. $n(\mathcal{G}_1)=3, n(\mathcal{G}_2)=12$.
Now, once we sum $\sum\limits_{z_1\cdots z_N}$ (10.3) the order of the points z_i will not anymore matter. We get finally the following graphical rules which we state for arbitary correlations :

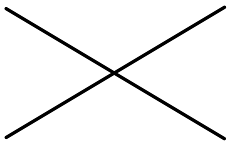
Definition A ϕ^4- graph with $2m$ *external legs*, labelled $1,\cdots,2m$, and N unlabelled *vertices* is a graph on N points (vertices) such that each vertex has 4 lines attached (one line can start and end at a vertex and is counted as two here), $2m$ lines have one end with no other lines attached.

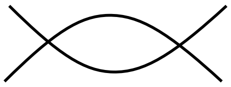
Examples


$2m=2,\ N=1$


$2m=2,\ N=2$


$2m=2,\ N=3$


$2m=4,\ N=1$


$2m=2,\ N=2$

With this notation, we have

$$\frac{d^n}{d\lambda^n}\,|_{\lambda=0} \int \prod_{i=1}^{2m} \phi(x_i) e^{-\lambda \sum_{x\in\Lambda}\phi(x)^4} d\mu_C(\phi) = (-1)^n \sum_{\mathcal{G}} n(\mathcal{G}) I(\mathcal{G})$$

where $\mathcal{G}$ run through $(2m,n)$-graphs, $n(\mathcal{G})$ is the number of pairings giving that $\mathcal{G}$ and $I(\mathcal{G})$ is the following expression :

1. Label the n vertices of $\mathcal{G}$ as $z_1,\cdots,z_n$
2. To each line $\ell=\{u,v\}$ of $\mathcal{G}$ (here u,v belong so $\{x_i\},\{z_j\}$ put $C_\ell = C(u-v)$.

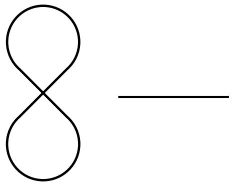
$$I(\mathcal{G}) = \sum_{z_1,\cdots,z_n\in\Lambda} \prod_\ell C_\ell. \tag{10.4}$$

We denote usually by

Example We get to $\mathcal{O}(\lambda)$:

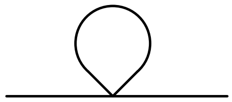
$$
\begin{aligned}
G_2 &= \\
&= \\
&= \\
&= C(x-y) - 12\lambda \sum_{z_1} C(x-z_1)C(z_1-z_1)C(z_1-y) + \mathcal{O}(\lambda^2) \\
&= C(x-y) - 12\lambda C(0) \sum_{z_1 \in \Lambda} C(x-z_1)C(z_1-y) + \mathcal{O}(\lambda^2)
\end{aligned}
$$

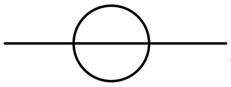
Remarks Note the presence of *disconnected* graphs like



that are cancelled upon normalization.

Homework : Prove :

$G_2 =$ $-\, 12\lambda$ 

$+\lambda^2(\alpha PICTURE + \beta PICTURE + \gamma$  $) + \mathcal{O}(\lambda^3$

and find α, β, γ

Theorem.

$$\frac{d^n}{d\lambda^n}\Big|_{\lambda=0} \frac{\int \phi(x)\phi(y) e^{-\lambda \sum \phi_x^4(x)} d\mu_C}{\int e^{-\lambda \sum \phi^4(x)} d\mu_C} = (-)^n \sum_{\mathcal{G}}{}^{c}\, n(\mathcal{G}) I(\mathcal{G})$$

where the sum $\sum^c$ runs over *connected* graphs.

Remark In the numerator the graphs are of the type

PICTURE

In the denominator

PICTURE

I.e. connected components have either both x and y or neither.

Note that disconnected graphs are here proportional to a power of the volume :

$$
\begin{aligned}
PICTURE &= C(x-y)\sum_{z\in\Lambda} C(0)^2 = |\Lambda| C(0)^2 C(x-y) \\
PICTURE &= C(x-y)C(0)^4|\Lambda|^2 \text{ etc}
\end{aligned}
$$

Proof of the Theorem Notation : $\langle F\rangle_\lambda = \frac{\int F(\phi)e^{-\lambda\sum\phi^4}d\mu_C}{\int e^{-\lambda\sum\phi^4}d\mu_C}$ so $\langle F\rangle_0 = \int F d\mu_C$.
Then, $\frac{d}{d\lambda}\langle F\rangle_\lambda = -\lambda\sum_x\langle\phi(x)^4F\rangle_\lambda + \lambda\sum\langle\phi(x)^4\rangle_\lambda\langle F\rangle_\lambda$ i.e. $\frac{d}{d\lambda}\langle F\rangle_\lambda = -\lambda\sum_x\langle\phi(x)^4;F\rangle_\lambda$ where we denote by

$$\langle F;G\rangle_\lambda = \langle FG\rangle_\lambda - \langle F\rangle_\lambda\langle G\rangle_\lambda$$

the so called *truncated* (or connected) correlation functions.

Lemma 1 Let $F = \prod_{\alpha\in A}\phi(x_\alpha), G = \prod_{\beta\in B}\phi(x_\beta)$.
Then $\langle F;G\rangle_0 = \sum_P\prod_{\langle\gamma\delta\rangle}C(x_\gamma - x_\delta)$ where each P has at least one pair $\langle\gamma\delta\rangle$ such that $\gamma\in A, \delta\in B$.

Proof Obvious, since $\langle F\rangle\langle G\rangle$ has those pairings where no such pairs occur and $\langle FG\rangle$ has all pairings
Now (Prove !)

$$\frac{d^n}{d\lambda^n}\mid_{\lambda=0}\langle F\rangle_\lambda = (-)^n\sum_{x_1\cdots x_n}\langle F;\phi(x_i)^4;\phi(x_2)^4;\cdots;\phi(x_n)^4\rangle_0$$

where

$$\langle F_1;F_2;\cdots;F_N\rangle = \sum_\pi(-1)^{|\pi|}\prod_\alpha\langle\prod_{i\in\pi_\alpha}F\rangle_0$$

where $\pi = \{\pi_\alpha\}_{\alpha=1}^{|\pi|}$ is a partition of $\{1,\cdots,N\}$ into $|\pi|$ subsets

<u>Ex</u> $\langle F_1;F_2;F_3\rangle = \langle F_1F_2F_3\rangle - \langle F_1F_2\rangle\langle F_3\rangle - \langle F_1F_3\rangle\langle F_2\rangle - \langle F_2F_3\rangle\langle F_1\rangle + \langle F_1\rangle\langle F_2\rangle\langle F_3\rangle$

Now the thing to check is

Lemma 2 Let $F_i = \prod_{\alpha\in A_i}\phi(x_{i_\alpha})$. Then

$$\langle F_1;\cdots;F_N\rangle_0 = \sum_P\prod C(x_\gamma - x_\delta)$$

where for each P there is a $\{\langle\gamma_i\delta_i\rangle\}_{i=1}^{N-1}\subset P$ that forms a conneted path, connecting the A_i's there exists a permutation ρ at the A_i's such that

$$\gamma_1\in A_{\rho(1)}, \delta_1\in A_{\rho(2)}, \gamma_2\in A_{\rho(2)}, \delta_2\in A_{\rho(3)}\cdots\delta_{N-1}\in A_{\rho(N)}.$$

PICTURE

The Lemma yields our theorem since there cannot be a connected graph with only one leg.

Problems 1. Prove that

$$G_4(x_1 \cdots x_4) \equiv \langle \prod_{i=1}^{4} \phi(x_i) \rangle$$

$$= G_2(x_1-x_2)G_2(x_3-x_4)+G_2(x_1-x_3)G_2(x_2-x_4)+G_2(x_1-x_4)G_2(x_2-x_3)+G_4^c(x_1,\cdots,x_4)$$

where $G_4^c = \sum^N \lambda^n G^c_{4,n} + \mathcal{O}(\lambda^{N+1})$ with $G^c_{4,n}$ consisting of connected graphs with 4 legs. Can you generalize to G_N?

2. Prove that $\frac{d^n}{d\lambda^n}|_0 \log Z$ is sum of connected vacuum graphs with n vertices. (a vacuum graph is a graph with no legs).

How to calculate a given graph? First, let us observe that in our model $G_{2m,n}$ has a $\Lambda \to \mathbf{Z}^d$ limit : Clearly it suffices to consider a connected graph which is an expression (let say $m = 1$)

$$\sum_{z_1 \cdots z_n \in \Lambda} \mathcal{G}(x_1, x_2, z_1 \cdots z_n)$$

and $\mathcal{G}$ is a product of $C(y_i - y_j)$ where the y's are x's or z's. Use $|C(x)| \leq Ae^{-|x|/\epsilon}$, and note that, since $\mathcal{G}$ is connected, there exists a connected tree graph in $\mathcal{G}$ containing all the vertices and all the legs (a tree graph has no loops)

Ex

PICTURE

Now do the z_i sums using $|\sum_{z_j \in \Lambda} C(z_i - z_j)| \leq$ const (independent on Λ) starting at ends of branches (e.g. z_3 above).

10.3 Momentum space representation

Actual calculations are easier in Fourier transform. Let $\Lambda = \mathbf{Z}^d$

Ex

$$PICTURE = \sum_{z_1 z_2} C(x - z_1)C(z_1 - z_2)^3 C(z_2 - y)$$

$$= \sum_{z_1 z_2} \int \hat{C}(p_1)\hat{C}(p_2)\hat{C}(p_3)\hat{C}(p_4)\hat{C}(p_5) e^{i[p_1(x-z_1)+p_2(z_1-z_2)+p_3(z_1-z_2)+ip_4(z_1-z_2)+ip_5(z_2-y)]} \prod \frac{dp_i}{(2\pi)^d}$$

$$= \int \prod \hat{C}(p_i)(2\pi)^d \delta(-p_1+p_2+p_3+p_4)(2\pi)^d \delta(-p_2-p_3-p_4+p_5) e^{ip_1x-ip_5y} \prod \frac{dp_i}{(2\pi)^d}$$

The first delta-function is due to the sum over z_1 and the second to the sum over z_2. Note “momentum conservation"at vertices.

PICTURE

$$= \int \frac{dp_1}{(2\pi)^d}\frac{dp_2}{(2\pi)^d}\frac{dp_3}{(2\pi)^d}\hat{C}(p_1)\hat{C}(p_2)\hat{C}(p_3)\hat{C}(p_1-p_2-p_3)\hat{C}(p_1)e^{ip_1(x-y)}$$

Thus PICTURE $= G(x-y) = \int \frac{dp}{(2\pi)^d}\hat{G}(p)e^{ip(x-y)}$

$$\hat{G}(p) = \left(\int \frac{dqdr}{(2\pi)^{2d}}\hat{C}(q)\hat{C}(r)\hat{C}(p-q-r)\right)\hat{C}(p)^2$$

or pictorially

DESSSIN

Note the momentum conservation and the integration over q, r. The general graph is now obvious :

PICTURE

1) But momentum p_i to legs, such that $\sum p_i = 0$
Reason : $G(x_1, \cdots, x_n)$ is translation invariant : $G(x_1, \cdots, x_n) = G(x_1+y, x_2+y, \cdots, x_n+y)$, $\forall y$, so it is a function of only $n-1$ variables, say $G(0, x_2-x_1, \cdots, x_n-x_1)$.

2) Satisfy momentum conservation at each vertex.

3) Integrate over each independent internal momentum $\frac{d^dp}{(2\pi)^d}$.

Ex

PICTURE

$$\begin{aligned}
&\hat{C}(p_1)\hat{C}(-p_1-p_2-p_3)\hat{C}(p_2)\hat{C}(p_3)\\
&\cdot\int\prod_{i=1}^{3}\frac{dq_i}{(2\pi)^d}\hat{C}(q_1)\hat{C}(q_2)\hat{C}(q_3)\\
&\cdot\hat{C}(p_1-q_1-q_2)\hat{C}(q_1+q_2-p_1-p_2-p_3)\\
&\cdot\hat{C}(p_1+p_2-q_1-q_2-q_3) \equiv \Gamma(p_1,p_2,p_3)
\end{aligned}$$

and in x space the graph is

$$G(x_1, \cdots, x_4) = \int e^{i(p_1(x_1-x_4)+p_2(x_2-x_4)+p_3(x_3-x_4))}\Gamma(p_1,p_2,p_3)\prod_{i=1}^{3}\frac{dp_i}{(2\pi)^d}$$

Remark 1 We have seen so far that :
a) If $|\Lambda| < \infty$, G_{2m} are C^∞ in λ

b) The Taylor coefficients $G^{(\Lambda)}_{2m,n}$ have a $\Lambda \to \mathbf{Z}^d$ limit.

It can be shown that G_{2m} have $\Lambda \to \mathbf{Z}^d$ limit, they are C^∞ in λ and Taylor coefficients are $\lim_{\Lambda\to\mathbf{Z}^d} G^{(\Lambda)}_{2m,n}$.

Remark 2 The expansion is not convergent. Consider e.g. a lattice consisting of one point i.e. the integral

$$F(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda\varphi^4 - \frac{r}{2}\varphi^2} d\varphi$$

Then

$$\alpha_n = \frac{d^n F}{d\lambda^n} |_{\lambda=0} = (-)^n \int \varphi^{4n} e^{-\frac{r}{2}\varphi^2} d\varphi = r^{-2n}(-1)^n \sqrt{2\pi}\frac{(4n)!}{2^{2n}(2n)!} \sim C^n (2n)!$$

as $n \to \infty$.
Thus the Taylor series $\sum \frac{1}{n!}\alpha_n \lambda^n$ has $\frac{\alpha_n}{n!} \sim C^n n!$ i.e. is very badly divergent.

Homework Prove that $F(\lambda)$ is analytic in the region $\mathbf{C}\backslash\{\lambda \in \mathbf{R}, \lambda \leq 0\}$

PICTURE

and has an essential singularity at $\lambda = 0$.

11 Infrared and Ultraviolet Divergencies

11.1 Infrared

Recall that for $\lambda = 0$, the critical point of our model is $r = 0$. What happens for $\lambda = 0$? The following is expected to be true :

For $\lambda = 0$, there exists r_c such that for $r > r_c$

$$0 \leq \langle \phi(x)\phi(y)\rangle \leq Ae^{-|x|/\xi} \quad , \xi < \infty \quad , \quad \xi(r) \underset{r \downarrow r_c}{\longrightarrow} \infty$$

For $r < r_c$ there are two phases, where $\langle \phi(x)\rangle = \pm m \neq 0$ and

$$|\langle \phi(x); \phi(y)\rangle | \leq Ae^{-|x|/\xi} \quad \xi < \infty,$$

with $\xi(r) \to \infty$ as $r \uparrow r_c$.
For $r = r_c$,

$$\lim_{|x-y|\to\infty} \langle \phi(x)\phi(y)\rangle |x-y|^{d-2+\eta} \neq 0$$

where η depends on the dimension d, $\eta = 0$ if $d \geq 4$, and equals the η of the Ising model.

These claims are proven, for small λ and $d \geq 4$. The rest remains a conjecture, although with plenty of theoretical and numerical evidence.

In this section we try to see what happens to our perturbation theory when $r = 0$. Let us consider the momentum space expressions for the graphs as $r = 0$. Since all integrals are over a bounded region $[-\pi, \pi]^d$ the only problem could come from $p = 0$ where $\mu(p) = 2\sum_{i=1}^{d}(1 - \cos p_i) = p^2 + \mathcal{O}(p^4)$ vanishes.

Consider the following graph

PICTURE

Now

$$PICTURE = \int_{[-\pi,\pi]^d} \frac{d^d q}{(2\pi)^d} \frac{1}{\mu(q)} \frac{1}{\mu(p-q)} \equiv I(p)$$

This is integrable in $d > 2$ if $p \neq 0$ and behaves as

$$I(p) \sim_{p \to 0} \begin{cases} |p|^{d-4} & d < 4 \\ \log|p| & d = 4 \\ \text{constant} & d > 4 \end{cases}$$

Thus PICTURE $= I(p)^n \sim |p|^{n(d-4)}$ which is not integrable if $n(4-d) \geq d$. (I.e. for $d = 3$ if $n \geq 3$). Hence e.g. the graph

PICTURE

has integral

$$\int \frac{I(q)^n}{\mu(p-q)} \frac{d^d q}{(2\pi)^d}$$

which is ill-defined for $d < 4$ and n large.

<u>Conclusion</u> The individual terms of our perturbation expansion

$$G_m = \sum \lambda^n G_{m,n}$$

$G_{m,n} = G_{m,n}(x_1 \cdots x_m, r)$ have no limit as $r \to 0$ if $d < 4$ (at least some of them). For $d \geq 4$ it may be shown that this limit exists. Hence we expect to need radically new ideas for $d < 4$. The $p \sim 0$ divergencies are called **IR** (infrared) divergencies.

11.2 Ultraviolet

Consider now field theory. We have the Gaussian measure $d\mu_G(\varphi)$, formally equal to

$$Z^{-1}\exp\left[-\frac{1}{2}\int[(\nabla\varphi)^2+\varphi(x)^2]d^dx\right]\prod_{x\in\mathbf{R}^d}d\varphi(x)$$

How about trying to construct the $\Lambda\to\mathbf{Z}^d$ limit of $e^{-\lambda\int_\Lambda\varphi(x)^4d^dx}d\mu_G(\varphi)$?
We immediately run into trouble : $\varphi(x)^4$, as we saw, is a bad random variable : $\int\varphi(x)^4d\mu_G=\infty$. Actually, we only know $\varphi(f)\equiv\int\varphi(x)f(x)dx$ for $f\in S(\mathbf{R}^d)$ is a measurable function on S' and has nice moments $\int\varphi(f)^nd\mu<\infty$, $\forall n$.

One way to proceed is to *regularize* the theory :
Recall : $\int\varphi(x)^4d\mu_G=3G(0)^2$ and $G(0)=\int_{\mathbf{R}^d}\hat{G}(p)\frac{d^dp}{(2\pi)^d}$. For us, $\hat{G}(p)=\frac{1}{p^2+m^2}$ so the divergence is due to insufficient decay of $\hat{G}(p)$ as $|p|\to\infty$ i.e. **UV** (ultraviolet) divergence. Thus, let us replace $\hat{G}(p)$ by

$$\hat{G}_\epsilon(p)=\hat{G}(p)\chi(\epsilon p)\equiv\hat{G}(p)\chi_\epsilon(p)$$

where $\chi\in S(\mathbf{R}^d),\chi(0)=1$. E.g. $\chi(p)=e^{-p^2}$ is fine. As $\epsilon\to0,\hat{G}_\epsilon(p)\to\hat{G}(p)$ pointwise. Now $\varphi(x)$ is a well defined random variable :

$$\int\varphi(x)^{2n}d\mu_{G_\epsilon}=\frac{(2n)!}{2^nn!}G_\epsilon(0)^n\quad,\quad G_\epsilon(0)=\int\hat{G}(p)\chi(\epsilon p)\frac{d^dp}{(2\pi)^d}<\infty$$

[To be pedantic : a priori we know only that $\varphi(f)$ is measurable, for $f\in S$, so consider $\varphi(f_{\epsilon,x}),f_{\epsilon,x}(y)=\dfrac{e^{-\frac{(x-y)^2}{2\epsilon}}}{(2\pi\epsilon)^{d/2}}$. This is a measurable function $\ell_\epsilon:S'\to\mathbf{R}:\varphi\to\varphi(f_{\epsilon,x})$. Thus $\lim_{\epsilon\to0}\ell_\epsilon$ also is measurable and this is what we call $\varphi(x)$ (formally : $\varphi(f_{\epsilon,x})=\int\varphi(y)f_{\epsilon,x}(y)dy\underset{\epsilon\to0}{\longrightarrow}\int\varphi(y)\delta(x-y)dy=\varphi(x)$)].
Hence, the measure

$$\frac{1}{Z_{\Lambda,\epsilon}}e^{-\int_\Lambda[a\varphi(x)^2+\lambda\varphi(x)^4]d^dx}d\mu_{G_\epsilon}(\varphi)\tag{11.1}$$

is well defined provided $a\in\mathbf{R},\lambda>0$.
[By Jensen, $Z\geq\exp\left[-|\Lambda|(aG_\epsilon(0)+3\lambda G_\epsilon(0)^2)\right]$ and since $a\varphi^2+\lambda\varphi^4\geq-\frac{a^2}{2\lambda}$, $Z\leq e^{-|\Lambda|\frac{a^2}{2\lambda}}$].
The correlation functions are again C^∞ in λ and a, and we have the expansion

$$G_{2m}=\sum_{n,p=0}^N\lambda^na^pG_{2m,n,p}+R_N\qquad R_N=\mathcal{O}(\lambda^{N+1},a^{N+1})$$

given in terms of graphs. The coefficients, $G_{2m,n,p}$ for $m^2>0$, have a $|\Lambda|\to\mathbf{R}^d$ limit but <u>not</u> an $\epsilon\to0$ limit, as we saw above.

The $\epsilon\to0$ limit question can be posed as follows.

Question Find three functions $Z(\epsilon), a(\epsilon), \lambda(\epsilon)$ such that

$$Z(\epsilon)^m \langle \prod_{i=1}^{2m} \varphi(x_i) \rangle_\epsilon$$

Converge, as $\epsilon \to 0$, to elements of $S'(\mathbf{R}^{2md})$ (or $Z(\epsilon)^m \langle \prod_{i=1}^{2m} \varphi(f_i) \rangle_\epsilon$, converge for all $f_i \in S(\mathbf{R}^d)$). Here $\langle\ \rangle_\epsilon$ is the expectation w.r.t. the measure (11.1), with $\Lambda = \mathbf{R}^d$, $a = a(\epsilon)$ and $\lambda = \lambda(\epsilon)$.

Note Obviously $\lambda(\epsilon) = 0, a(\epsilon) = 0, Z = 1$ is a solution. We ask if there are any limits that are *non-Gaussians*. The answer is pretty well known :
For $d = 2, Z = 1, a = -3\lambda G_\epsilon(0), \lambda(\epsilon) = \lambda$ works.
For $d = 3, Z = 1, a = -3\lambda G_\epsilon(0) + \alpha\lambda^2 \log \epsilon$ (α explicit), $\lambda(\epsilon) = \lambda$ works.
For $d \geq 4$ no non-Gaussian limit.

Our objective is to explain these observations.

Remark χ_ϵ is a regulator. It cuts off momentum $|p| \gtrsim \frac{1}{\epsilon}$ very effectively. We could also use a lattice-cutoff. I.e. consider the following Hamiltonian on $(\epsilon\mathbf{Z})^d : \varphi : (\epsilon\mathbf{Z})^d \to \mathbf{R}$:

$$\mathcal{H} = \sum_{|x-y|=\epsilon} \frac{(\varphi(x) - \varphi(y))^2}{\epsilon^2} \epsilon^d + \frac{r}{2} \sum \epsilon^d \varphi(x)^2 + \sum \epsilon^d (a\varphi(x)^2 + \lambda\varphi(x)^4) \tag{11.2}$$

and study

$$Z(\epsilon)^m \langle \prod_{i=1}^{2m} \varphi(x_i) \rangle_\epsilon$$

As we saw before, this equals, by putting

$$\phi(x) = \epsilon^{\frac{d-2}{2}} \varphi(\epsilon x) \quad , \quad x \in \mathbf{Z}^d$$

$$Z(\epsilon)^m \epsilon^{(2-d)m} \langle \prod_{i=1}^{2m} \phi(\frac{x_i}{\epsilon}) \rangle^{(\epsilon)}$$

where $\langle\ \rangle^{(\epsilon)}$ is the expectation w.r.t. the measure

$$\lim_{\Lambda \to \mathbf{Z}^d} \frac{1}{Z_{\Lambda,\epsilon}} \exp\left[-\sum_{x \in \Lambda} \epsilon^2 a\phi(x)^2 + \epsilon^{4-d}\lambda\phi(x)^4\right] d\mu_{C_\epsilon}(\phi)$$

where $C_\epsilon = \frac{1}{-\Delta + \epsilon^2 r}$ on $\ell^2(\mathbf{Z}^d)$.
Thus our **UV** problem $\epsilon \to 0$ is the same thing as staying on a fixed lattice, taking distances to ∞ (x_i/ϵ), going to critical point $(\epsilon^2 r \to 0)$. It should now be obvious that both **IR** and **UV** problem have something to do with Statistical Mechanics at the critical point.

Remark The ϵ-lattice theory can be interpreted as a particular cutoff of the continuum theory. Namely, the measure with $\mathcal{H}$ as in (11.2) with $a = \lambda = 0$ is the Gaussian measure on $\mathbf{s}'((\epsilon\mathbf{Z})^d)$ with covariance $C_\epsilon = (-\Delta_\epsilon + r)^{-1}$ where the latter is an operator on $\ell^2(\epsilon\mathbf{Z}^d)$ i.e. $(-\Delta_\epsilon\varphi)(x) = \sum_{|i|=1}[\varphi(x) - \varphi(x + \epsilon i)]\epsilon^{-2}$ for $\varphi : (\epsilon\mathbf{Z})^d \to \mathbf{R}$. Concretely

$$\frac{1}{-\Delta_\epsilon + r}(x, y) = \int_{[-\frac{\pi}{\epsilon},\frac{\pi}{\epsilon}]^d} \frac{e^{ip(x-y)}}{\epsilon^{-2}\mu(\epsilon p) + r} \frac{d^d p}{(2\pi)^d}$$

which shows that p is cut-off to $|p_i| \leq \pi/\epsilon$. Similarily, instead of the lattice model we may consider the Statistical Mechanics where φ is on $\mathbf{R}^d$ but has the cutoff χ, i.e. the measure

$$\lim_{\Lambda\to\mathbf{Z}^d} \frac{1}{Z_\Lambda} e^{-\lambda\int_\Lambda \varphi(x)^4 dx} d\mu_{G_1}(\varphi)$$

with $G_1(p) = \frac{1}{p^2+r}\chi(p)$.

12 The Renormalization Group

12.1 The Block-spin Transformation

Suppose that we have a Statistical Mechanics model at its critical point such that

$$\left| \langle\phi(x)\phi(y)\rangle - \frac{A}{|x-y|^a} \right| \leq \frac{B}{|x-y|^{a+\epsilon}} \tag{12.3}$$

(as $|x - y| > 1$) where $\epsilon > 0$. Then, consider the following transformation : $\phi \in \mathbf{R}^{\mathbf{Z}^d} \to \phi_L \in \mathbf{R}^{\mathbf{Z}^d}$:

$$\phi_L(x) = L^{\frac{a}{2}-d} \sum_{y:|y_i|<\frac{L}{2}} \phi(Lx + y) \tag{12.4}$$

Here $L > 1$ is odd and the sum is over the L-cube at Lx :

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet\, Lx & \bullet \\ \bullet & \bullet & \bullet\, Lx+y \end{array} \qquad \text{Here } L = 3$$

Note : $L^{-d}\sum\phi(Lx + y)$ is just the *average* of ϕ in L-cube centered at Lx. We see that $\phi \to \phi_L$ involves :

a) "Coarse graining": average over details of ϕ for scales $\leq L$.

b) Scaling : $\phi_L(x)$ depends on ϕ near Lx and we multiply ϕ by $L^{a/2}$

$\phi \to \phi_L$ is called the *Block-spin transformation.*
(12.4) defines a linear map $C_L : \mathbf{R}^{\mathbf{Z}^d} \to \mathbf{R}^{\mathbf{Z}^d}$, concretely,

$$(C_L)_{xy} = L^{\frac{a}{2}-d} \begin{cases} 1 & |(Lx - y)_i| < L/2 \\ 0 & \text{otherwise} \end{cases}$$

Note that

$$C_L^2 = C_{L^2}$$

(Prove !). Thus, the iteration of (12.4) n times is the same as doing it once with L^n :

$$\begin{aligned} \phi_{L^n}(x) &= (C_{L^n}\phi)(x) = (C_L^n\phi)(x) \\ &= L^{n(\frac{a}{2}-d)} \sum_{|y_i|<L^n/2} \phi(L^n x + y) \end{aligned}$$

Now observe :

Proposition Suppose that (12.3) holds. Then, for $\max_i |x_i - y_j| > 1$,

$$\langle (C_L^n\phi)(x)(C_L^n\phi)(y)\rangle_{\overrightarrow{n\to\infty}} AG^*(x,y)$$

with

$$G^*(x,y) = \int_{du} \int_{dv |x-y+u-v|^{-a}}$$

and this holds also for all x, y, if $a < d$. Here $= [-\frac{1}{2}, \frac{1}{2}]^d$.

<u>Remark</u> G^* decays as $\frac{1}{|x-y|^a}$ as $|x-y| \to \infty$.

<u>Proof</u> We get, for $x \neq y$,

$$\begin{aligned} &| \langle C_L^n\phi(x) C_L^n\phi(y)\rangle - AL^{(a-2d)n} \sum_{\substack{u,v \\ |v_i|,|u_i|<L^n/2}} |L^n(x-y)+u-v|^{-a} \;| \\ &\leq BL^{(a-2d)n} \sum_{u,v} |L^n(x-y)+u-v|^{-a-\epsilon} \\ &= BL^{-n\epsilon} \sum_{u,v} L^{-2dn} |x-y+L^{-n}(u-v)|^{a-\epsilon} \\ &\overrightarrow{n\to\infty} 0 \cdot \int_{du} \int_{dv |x-y+u-v|^{-a-\epsilon} = 0} \end{aligned}$$

and the A-term $\to A \int_{du} \int_{dv|x-y+u-v|^{-a}}$. We used $\max |x_i - y_i| > 1$, $|(u-v)_i| < 1 \Rightarrow |x-y+u-v| > 0$, so integrals converge. The rest is similar

<u>Example</u> For the Gaussian Ginzburg-Landau model with covariance $\frac{1}{-\Delta}$ we take $a = d-2$ and get

$$\begin{aligned} \langle C_L^n\phi(x) C_n^n\phi(y)\rangle &= L^{n(d-2)} L^{-2nd} \sum_{u,v} \int_{[-\pi,\pi]^d} \frac{e^{ip(L^n x - L^n y + u - v)}}{\mu(p)} \\ &= \sum_{u,v} L^{-2n} \int_{[-L^n\pi, L^n\pi]^d} \frac{d^d p}{(2\pi)^d} \frac{e^{ip(x-y+L^{-n}(u-v))}}{\mu(L^{-n}p)} \\ &\overrightarrow{n\to\infty} \int_{du} \int_{dv} \left(\frac{1}{-\Delta_{\mathbf{R}^d}}\right)(x-y+u-v) \end{aligned}$$

and $-\Delta_{\mathbf{R}^d}$ is the usual $-\Delta$ on $L^2(\mathbf{R}^d)$. Now,

$$\frac{1}{-\Delta_{\mathbf{R}^d}}(x-y) = \frac{\text{const}}{|x-y|^{d-2}}.$$

Remark In this example, $C_L^n \phi(x)$ are Gaussian, with covariance $C_L^n \frac{1}{-\Delta}(C_L^n)^T$. T denotes the transpose, this is just above, i.e.

$$\sum_{z,w} (C_L^n)_{xz} (C_L^n)_{yw} \left(\frac{1}{-\Delta}\right)_{zw}$$

How about generally?

12.2 Transformations on measures

Suppose μ is a cylinder measure in $\mathbf{R}^{\mathbf{Z}^d}$ and consider the random variables $C_L\phi$. Their generating function is

$$\begin{aligned} W_L(f) &= \int e^{i(C_L\phi, f)} d\mu(\phi) = \int e^{i(\phi, C_L^T f)} d\mu(\phi) \\ &= W(C_L^T f) \end{aligned}$$

which is obviously of positive type, so there is a measure μ_L such that

$$W_L(f) = \int e^{i(\Psi, f)} d\mu_L(\Psi).$$

μ_L is the probability distribution of the block spins $C_L\phi$. Thus $C_L : \mathbf{R}^{\mathbf{Z}^d} \to \mathbf{R}^{\mathbf{Z}^d}$ induced a map

$$C_L^* : \mathcal{B} \to \mathcal{B}$$

where $\mathcal{B}$ are the cylinder measures on $\mathbf{s}'(\mathbf{Z}^d)$.
C_L^* is called the *Renormalization Group Transformation* on measures.

12.3 Transformations on Hamiltonians

Concretely, let us work in finite volume Λ_N and take $\Lambda_N = L^N$-box centered at the origin. Thus $\phi \in \mathbf{R}^{\Lambda_N}, C_L\phi \in \mathbf{R}^{\Lambda_{N-1}}$ and $C_L^n\phi \in \mathbf{R}^{\Lambda_{N-n}}$. Let μ be a probability measure

$$d\mu(\phi) = \frac{1}{Z} e^{-\mathcal{H}(\phi)} d^{\Lambda_N}\phi$$

when $\mathcal{H}$ is some function $\mathcal{H} : \mathbf{R}^{\Lambda_N} \to \mathbf{R}$ such that $e^{-\mathcal{H}}$ is integrable. Then,

$$d\mu_L(\Psi) = F(\Psi) d^{\Lambda_{N-1}}\Psi$$

with

$$F(\Psi) = \frac{1}{Z} \int e^{-\mathcal{H}(\phi)} \prod_{x \in \Lambda_{N-1}} \delta(\Psi(x) - (C_L\phi)(x)) d^{\Lambda_N}\phi$$

(Z normalizes $\int F d^{\Lambda_{N-1}}\Psi = 1$).
Let us write this more explicitely : $\Psi(x)$ equals $L^{\frac{a}{2}}$ times the average of ϕ in the L-cube centered at Lx. Thus, if we fix $\Psi(x), \forall x \in \Lambda_{N-1}$, we integrate out all fluctuations around this average. In other words, let ϕ_0 be a configuration that is constant in each L-cube, and equals the average of ϕ there. I.e. $\phi_0(Lx+y) = L^{-\frac{a}{2}}\Psi(x)$ for all $x \in \Lambda_{N-1}$, all y with $|y_i| < L/2$. Using our previous notation this equals

$$\phi_0 = L^{-\frac{a}{2}} L^{-\frac{a}{2}+d} C^T \Psi = L^{d-a} C^T \Psi$$

(recall, $C^T_{xy} = L^{\frac{a}{2}-d}$ if $x \in L$-cube at $Ly, 0$ otherwise). Thus $\phi = \phi_0 + Z$ with Z having 0 average over L-cubes i.e. $C_L Z = 0$. Thus

$$F(\Psi) = \frac{1}{Z} \int e^{-\mathcal{H}(L^{d-a}C^T\Psi + Z)} \prod_{x \in \Lambda_{N-1}} \delta\Big((C_L Z)(x)\Big) d^{\Lambda_N} Z$$

Even more concretely, take $z \in \mathbf{R}^{\Lambda_N \backslash L\Lambda_{N-1}}$ i.e. $z(x) \in \mathbf{R}, \forall x \in \Lambda_N$ where $x \neq Lw$, $w \in \mathbf{Z}^d$. Put

$$Z(x) = \begin{cases} z(x) & x \in \Lambda_N \backslash L\Lambda_{N-1} \\ -\sum_{|y_i|<L/2} z(Lw+y) & x = Lw \end{cases}$$

Then $C_L Z = 0$ and so

$$F(\Psi) = \frac{1}{Z} \int e^{-\mathcal{H}(L^{d-a}C^T\Psi + Z)} d^{\Lambda_N \backslash L\Lambda_{N-1}} z$$

a concrete integral over "fluctuation variables"$z(x)$.
Let $F = e^{-\mathcal{H}'}$. The map $R_L : \mathcal{H} \to \mathcal{H}'$ is called the *RG Transformation on Hamiltonians*, i.e.

$$R_L \mathcal{H} = -\log \frac{1}{Z} \int e^{-\mathcal{H}(L^{d-a}C^T\Psi + Z)} dz \tag{12.5}$$

Here $\mathcal{H} : \mathbf{R}^{\Lambda_N} \to \mathbf{R}$ and $R_L\mathcal{H} : \mathbf{R}^{\Lambda_{N-1}} \to \mathbf{R}$. Note how the volume contracted from an L^N-box to an L^{N-1}-box.

Remark 1. It was trivial to define the RG on measures and in infinite volume : $C_L^*\mu$ is automatically a measure. On Hamiltonians, we need to
a) work in finite volume,
b) make some assumptions on $\mathcal{H}$ so that $R_L\mathcal{H}$ (12.5) is well defined.

2. RG is called "group"because $C_L^n = C_{L^n}$ i.e. $(C_L^*)^n = C_{L^n}^*$ and $(R_L)^n = R_{L^n}$. Actually, it is a semigroup : $R_1 = \text{id}$ but R_L^{-1} is not defined.
Our Proposition above says that

$$\int (C_L^n\phi)(x)(C_L^n\phi)(y) d\mu \underset{n\to\infty}{\longrightarrow} G^*(x,y)$$

(absorbing the constant A in the definition of G^*), i.e.

$$\int \Psi(x)\Psi(y) d\mu_{L^n} \to G^*(x,y)$$

Thus, we might hope that the measures μ_{L^n} converge to some measure μ^* such that

$$\int \Psi(x)\Psi(y)d\mu^*(\Psi) = G^*(x,y)$$

and μ^* is a *fixed point* of C_L^* :

$$C_L^*\mu^* = \mu^*$$

Equivalently, provided we make sense of $\mathcal{H}$ in infinite volume, we might expect

$$(R_L)^n\mathcal{H} \underset{n\to\infty}{\longrightarrow} \mathcal{H}^*, \qquad R_L\mathcal{H}^* = \mathcal{H}^*$$

in some topology in a space of $\mathcal{H}$'s.

<u>Remark</u> It is actually quite difficult to set up nice spaces of Hamiltonians for unbounded spins : things are very sensitive to the dependence of $\mathcal{H}$ on large values of $\phi(x)$'s : we need the $\phi(x)$ integral to converge at $\pm\infty$. This problem did not occur for bounded spins where we could define $\mathcal{H}$ via potentials Φ_X and talk about the Banach space $\mathcal{B}$ of Hamiltonians (i.e. of Φ_X's) : Given $\Phi \in \mathcal{B}$, we could define the set of Gibbs measures μ in infinite volume corresponding to Φ. Moreover, above the critical temperature, we expected Φ to uniquely determine a μ. For unbounded spins no similar formalism exists.

For bounded spins σ the RG, as defined above, will in general change the range of values that σ takes : e.g. for Ising spins $\sigma(x) \in \{\pm 1\}$ $C_L\sigma$ takes values in the set $\{\sigma \in L^{\frac{a}{2}-d}\mathbf{Z} | |\sigma| \leq L^{\frac{a}{2}}\}$ i.e. for $a > 0$, larger values. Upon iteration, $C_L^n\sigma(x) \in L^{(\frac{a}{2}-d)n}\mathbf{Z}$, $|C_L^n\sigma(x)| \leq L^{\frac{an}{2}}$. Thus, e.g. at $d = 3$, where we expect $a = d - 2 + \eta, \eta \approx 0.1$, as $n \to \infty$ the block spins become *unbounded* and continuous! Anyway, for $n < \infty$, we have a bounded spin model and can ask whether R_L actually maps Hamiltonians to Hamiltonians. I.e. suppose μ is a Φ-Gibbs measure, $\Phi \in \mathcal{B}$. Is $C_L^*\mu$ a Gibbs measure for some Φ'? If so, define $R_L\Phi = \Phi'$. This is expected to be true in Ising model, but proven only for small β or large β (where one needs to enlarge $\mathcal{B}$, surprisingly!).

For unbounded spins, R_L and its iterations have been rigorously studied for the Ginzburg-Landau model. It turns out that one can, to a certain extent set up a space of Hamiltonians where R_L acts; however, one has to supplement this with a different representation of μ for large values of ϕ's.

For the rest of the time, we will not discuss these problems. Rather we

a) Pretend that R_L is defined in some space of $\mathcal{H}$'s and see what implications this could have (Section 13).

b) Carry out a perturbative analysis of $R_L^n\mathcal{H}$ for $\mathcal{H}$ = Ginzburg-Landau model and understand the **IR** and **UV** problem this way (Sections 14-17).

13 R_L near a Fixed Point : Critical Exponents

13.1 General Framework

We assume the following setup

1. We have some space $\mathcal{K}$ of "Hamiltonians"$\mathcal{H}$ such that $\mathcal{H} \in \mathcal{K}$ determines a unique measure $\mu_{\mathcal{H}}$ on $\mathbf{R}^{\mathbf{Z}^d}$. We denote $\langle F \rangle_{\mathcal{H}} = \int F d\mu_{\mathcal{H}}$. Here $\mu_{\mathcal{H}}$ is the limit of $\frac{1}{Z_\Lambda} e^{-\mathcal{H}_\Lambda} \phi$ as $\Lambda \nearrow \mathbf{Z}^d$ and $\mathcal{H}_\Lambda$ is some finite volume version of $\mathcal{H}$. (or more properly, $\mathcal{H} = \{\mathcal{H}_\Lambda\}_{\Lambda \subset \mathbf{Z}^d}$).

2. The RG is defined in $\mathcal{K}$: $R_L : \mathcal{K} \to \mathcal{K}$ i.e.

$$C_L^* \mu_{\mathcal{H}} = \mu_{R_L \mathcal{H}}$$

We also suppose that there is a metric in $\mathcal{K}$ and if $\mathcal{H}_n \to \mathcal{H}$, then $\mu_{\mathcal{H}_n} \to \mu_{\mathcal{H}}$.
Let us see what kind of picture of critical phenomena emerges from such assumptions.
Let $\mathcal{H}^*$ be a fixed point of R_L : $R_L \mathcal{H}^* = \mathcal{H}^*$.

Definition The stable manifold of $\mathcal{H}^*$ in $\mathcal{K}$ is

$$\mathcal{M}_s = \{\mathcal{H} | R_L^n \mathcal{H} \to \mathcal{H}^*\}$$

What can we say about the decay of correlations of $\mathcal{H} \in \mathcal{M}_s$? This depends on a in the definition of C_L : $(C_L \phi)(x) = L^{\frac{a}{2}} L^{-d} \sum_{|y_i| < \frac{L}{2}} \phi(Lx + y)$. Recall that if $\langle \phi(x) \phi(y) \rangle_{\mathcal{H}}$ decays like $|x - y|^{-a}$ then $\langle \phi(x) \phi(y) \rangle_{\mathcal{H}^*}$ does too. We suppose $|\langle \phi(x) \phi(y) \rangle_{\mathcal{H}^*}| < \infty$ $\forall x, y$ and $\exists x_0, y_0$ $\langle \phi(x_0) \phi(y_0) \rangle_{\mathcal{H}^*} \neq 0$.

Proposition a) Suppose $a < d$. Then all $\mathcal{H} \in \mathcal{M}_s$ are *critical*, i.e. for no $A > 0$, $\xi < \infty$ $|\langle \phi(x) \phi(y) \rangle_{\mathcal{H}}| \leq A e^{-|x-y|/\xi}$ for all x, y.
b) If $a = d$ and $\exists \mathcal{H} \in \mathcal{M}_s$ which is not critical, then $\langle \phi(x) \phi(y) \rangle_{\mathcal{H}^*} = 0$ for $x \neq y$.

Proof a) Suppose $\exists A, \xi$. Then,

$$\begin{aligned}
0 \neq |\langle \phi(x_0) \phi(y_0) \rangle_{\mathcal{H}^*}| &= |\lim_{n \to \infty} \langle \phi(x_0) \phi(y_0) \rangle_{R^n \mathcal{H}}| \\
&= |\lim_{n \to \infty} \langle C_L^n \phi(x_0) C_L^n \phi(y_0) \rangle_{\mathcal{H}}| \\
&= |\lim_{n \to \infty} L^{n(a-2d)} \sum_{|u_i| < L^n/2} \sum_{|v_i| < L^n/2} \langle \phi(L^n x_0 + u) \phi(L^n y_0 + v) \rangle_{\mathcal{H}} \\
&\leq A \lim_{n \to \infty} L^{n(a-2d)} \sum \exp[-L^n |x_0 - y_0 + L^{-n} |u - v|]/\xi \\
&= A \lim_{n \to \infty} L^{na} \int_{du} \int_{dv} e^{-L^n |x_0 - y_0 + u - v]/\xi} \\
&\leq AC \lim_{n \to \infty} L^{n(a-d)} = 0 \Rightarrow \text{ contradiction}
\end{aligned}$$

b) Now, if $x_0 \neq y_0$ this is $\leq AC \lim L^{n(a-d-1)} = 0$.

Remark Also, if $\langle\phi(x_0)\phi(y_0)\rangle_{\mathcal{H}^*} \neq 0$ for $|x_0 - y_0| > 1$ get above $\leq \lim L^{na} e^{-L^n} = 0$.
Thus, if $a < d$, both $\mathcal{H}$ and $\mathcal{H}^*$ are critical.
If $a = d$ our block spins are

$$\phi_L(x) = L^{-\frac{d}{2}} \sum \phi(Lx + y)$$

i.e. we normalize like independent random variables. Above calculation just showed that if $\phi(x)$'s are (exponentially) weakly dependent then $\phi_{L^n}(x)$ become *independent* as $n \to \infty$ (central limit theorem).
Thus $a < d$ is the interesting case.
[By the way, it can be proven quite generally that if $\langle\phi(x)\phi(y)\rangle$ decays as $|x-y|^{-a}$, $a > d$ then it decays exponentially].
We can actually argue that, for all $\mathcal{H} \in \mathcal{M}_s$,

$$\langle\phi(x)\phi(y)\rangle_{\mathcal{H}} \sim |x-y|^{-a} \quad \text{as } |x-y| \to \infty$$

Indeed, from above,

$$L^{(a-2d)n} \sum_{|u_i| < L^n/2} \quad \sum_{|v_i| < L^n/2} \langle\phi(L^n x + u)\phi(L^n x + v)\rangle \underset{n\to\infty}{\longrightarrow} \langle\phi(x)\phi(y)\rangle_{\mathcal{H}^*}$$

and RHS $\neq 0$ for infinitely many x, y (otherwise $\mathcal{H}^*$ has $\xi < \infty \Rightarrow$ contradiction). Fix such x, y with $|x-y| \geq 2$. Then in LHS $|L^n x + u - L^n y - v| \geq \frac{1}{2} L^n |x-y|$ so there have to be many $\tilde{x}, \tilde{y}$ with $|\tilde{x} - \tilde{y}| \sim L^n$ such that $\langle\phi(\tilde{x})\phi(\tilde{y})\rangle \sim L^{-an}$, which is our claim

Summary $\mathcal{M}_s$ is a *critical surface* consisting of $\mathcal{H} \in \mathcal{K}$, all having the same critical exponent for the 2-point function.

Next, consider

$$\mathcal{M}_\xi = \{\mathcal{H} | \mathcal{H} \text{ has correlation length } \xi\}$$

For this we assume that all $\mathcal{H} \in K$ have a well defined ξ, i.e. the limit

$$\lim_{|x|\to\infty} -\frac{1}{|x|} \log\langle\phi(0)\phi(x)\rangle = \xi \text{ exists}$$

Then a) $\mathcal{M}_s = \mathcal{M}_\infty$

b) $R_L : \mathcal{M}_\xi \to \mathcal{M}_{\xi/L}$

b) follows since R_L scales by L:

$$\langle\phi(0)\phi(x)\rangle_{R_L\mathcal{H}} = L^\alpha \sum_{u,v} \langle\phi(u)\phi(Lx+v)\rangle L^{-2d} \sim e^{-L|x|/\xi}.$$

Thus, we have a picture of $\mathcal{K}$:

PICTURE
R_L takes :
- critical $\mathcal{H}$ to $\mathcal{H}^*$ upon iteration
- noncritical $\mathcal{H}$ away from $\mathcal{H}^*$ upon iteration.

13.2 Linear Analysis around Fixed Points

Let us study R_L near $\mathcal{H}^*$.
We assume that the usual dynamical systems ideas are applicable i.e. that R_L is, say, a smooth map in $\mathcal{K}$[6]. Thus, let $\mathcal{L} = DR_{\mathcal{H}^*}$ be the derivative of R at $\mathcal{H}^*$. This means that, for any $\mathcal{H} \in \mathcal{K}$,

$$R(\mathcal{H}^* + \epsilon\mathcal{H}) = \mathcal{H}^* + \epsilon\mathcal{L}\mathcal{H} + \mathcal{O}(\epsilon^2)$$

The spectrum of $\mathcal{L}$ plays now a crucial rule in the analysis of R near $\mathcal{H}^*$.
In the known examples, the spectrum of $\mathcal{L}$ has the following structure :

a) It is discrete, consisting of, real, positive, eigenvalues λ_i of finite multiplicity

b) There is a finite number of λ_i with $\lambda_i \geq 1$.

Let $\mathcal{H}_i$ be an eigenvector : $\mathcal{L}\mathcal{H}_i = \lambda_i\mathcal{H}_i$. We say that :

1° $\mathcal{H}_i$ is *relevant* if $\lambda_i > 1$

2° $\mathcal{H}_i$ is *marginal* if $\lambda_i = 1$

3° $\mathcal{H}_i$ is *irrelevant* if $\lambda_i < 1$

Note that $DR^n_{\mathcal{H}^*} = D(R \circ R \circ R \cdots R)_{\mathcal{H}^*} = (DR_{\mathcal{H}^*})^n = \mathcal{L}^n$ (since $R(\mathcal{H}^*) = \mathcal{H}^*$), so

$$R^n(\mathcal{H}^* + \epsilon\mathcal{H}_i) = \mathcal{H}^* + \epsilon\lambda_i^n\mathcal{H}_i$$

i.e. $\mathcal{H}_i$ perturbation increases ($\lambda_i > 1$) or decreases ($\lambda_i < 1$) exponentially in n or stays constant ($\lambda_i = 1$). Let us interpret this. Suppose first the case where there is only one $\lambda > 1$, with multiplicity 1, say λ_1. Then for $|x-y| >> 1$

$$\langle\phi(L^n x)\phi(L^n y)\rangle_{\mathcal{H}^*+\epsilon\mathcal{H}_1} \approx L^{-na}\langle C_L^n\phi(x)C_L^n\phi(y)\rangle_{\mathcal{H}^*+\epsilon\mathcal{H}_1}$$
$$= L^{-na}\langle\phi(x)\phi(y)\rangle_{R^n(\mathcal{H}^*+\epsilon\mathcal{H})} = L^{-na}\langle\phi(x)\phi(y)\rangle_{\mathcal{H}^*+\epsilon\lambda_1^n\mathcal{H}_1+\mathcal{O}(\epsilon\lambda_1^n)^2}$$

[6] If $\mathcal{K}$ is a Banach space, the derivative of R near $\mathcal{H}^*$ at $\mathcal{H}$ is defined as usual as the linear map $DR_{\mathcal{H}}$ $: \mathcal{K} \to \mathcal{K}$:

$$DR_{\mathcal{H}}\mathcal{H}_1 = \lim_{\epsilon\to 0}\frac{1}{\epsilon}[R(\mathcal{H}+\epsilon\mathcal{H}_1) - R(\mathcal{H})]$$

.

Take n such that $\epsilon\lambda_1^n = \mathcal{O}(1)$. Recall that we had one direction in $\mathcal{K}$ where we *depart* from $\mathcal{H}^*$ upon iteration, namely the one parametrized by ξ. Thus, it is natural to assume that $\mathcal{H}^* + \epsilon\mathcal{H}_1$ has $\xi < \infty$ and $\mathcal{H}^* + \mathcal{O}(1)\mathcal{H}_1$ has $\xi = \mathcal{O}(1)$. From above, then

$$\begin{aligned} \xi(\mathcal{H}^* + \epsilon\mathcal{H}_1) &= L^{+n}\xi(\mathcal{H}^* + \mathcal{O}(1)\mathcal{H}_1) \\ &= L^n\mathcal{O}(1) \end{aligned}$$

Write $\lambda_1 = L^\alpha$ (so $\alpha > 0$). Then $\epsilon \sim L^{-n\alpha} \Rightarrow \xi(\mathcal{H}^* + \epsilon\mathcal{H}_1) \sim \epsilon^{-1/\alpha}$
Let us identify $\epsilon = 0$ with $T = T_c$ (T = temperature). Thus, our Hamiltonian $\mathcal{H}^*$ could be say Ising model $\beta_c\mathcal{H}_{Ising}$ and $\mathcal{H}^* + \epsilon\mathcal{H}_1$ would be $\beta\mathcal{H}_{Ising}$ with $(1+\epsilon)\beta^* = \beta$ or $\epsilon \sim T - T_c$. More generally, we could start with Ising and consider some effective long distance model (see below for precise definitions) for its correlations. Then again $\epsilon \sim T - T_c$. So then

$$\xi \sim (T - T_c)^{-1/\alpha}$$

and $-1/\alpha$ is the so called correlation length critical exponent describing how ξ diverges as $T \to T_c$[7].

Universality Consider now $\mathcal{H} = \mathcal{H}^* + \epsilon\mathcal{H}_i, \lambda_i < 0$. As before,

$$\begin{aligned} L^{na}\langle C_L^n\phi(x)C_L^n\phi(y)\rangle_{\mathcal{H}} &= \langle\phi(x)\phi(y)\rangle_{\mathcal{H}^*+\epsilon\lambda_i^n\mathcal{H}_i} \\ &= \langle\phi(x)\phi(y)\rangle_{\mathcal{H}^*} + \mathcal{O}(\lambda_i^n) \end{aligned}$$

so, if $\lambda_i = L^{-\alpha_i}$, $\alpha_i > 0$, then, since $|u-v| \sim L^n \Rightarrow L^{-\alpha_i n} \sim |u-v|^{\alpha_i}$. Thus these directions give subleading corrections to the decay determined by $\mathcal{H}^*$. We have *universality* : The leading assymptotics of $\langle\prod_{i=1}^N \phi(L^n x_i)\rangle_{\mathcal{H}}$ is $\langle\prod_{i=1}^N \phi(L^n x_i)\rangle_{\mathcal{H}^*}$ if $\mathcal{H} \in \mathcal{M}_s$ (as $n \to \infty$ and x_i are apart from each other) and the details of $\mathcal{H}$ are only seen in the $\mathcal{O}(L^{-\alpha_i n})$ corrections. We say that all $\mathcal{H} \in \mathcal{M}_s$ <u>have the same critical behaviour</u>.
Also, all the exponents α_i are determined by $\mathcal{H}^*$, which appear not to depend on which $\mathcal{H}$ we have (in $\mathcal{M}_s$).

14 The Gaussian Fixed Point

14.1 Definition of RG in Momentum Space

As we saw above, the RG drives upon iteration the Gaussian measure μ_C where $C = \frac{1}{-\Delta}$ (on $\mathbf{Z}^d$) to the fixed point

$$C^*(x-y) = \int_{du} \int_{dv} \int_{\mathbf{R}^d} e^{ip(x-y)}\, e^{ip(u-v)}\, \frac{1}{p^2} \frac{d^dp}{(2\pi)^d} \qquad x, y \in \mathbf{Z}^d$$

[7]Note that we got $\langle\phi(x)\phi(y)\rangle_T \sim \frac{1}{|x-y|^a} e^{-\frac{|x-y|}{\xi(T)}}$ for all x, y.

i.e.

$$\lim_{n\to\infty}\langle C_L^n\phi(x)C_L^n\phi(y)\rangle_{\mu_C}=C^*(x-y).$$

Since

$$\int_{du e^{ipu}}=\prod_{\mu=1}^d\int_{-1/2}^{1/2}e^{ip_\mu u_\mu}du_\mu=\prod_{\mu=1}^d\frac{2\sin p_\mu/2}{p_\mu},$$

we have,

$$C^*(x-y)=\int_{\mathbf{R}^d}e^{ip(x-y)}\frac{1}{p^2}\prod_{\mu=1}^d\left(\frac{2\sin p_\mu/2}{p_\mu}\right)^2\frac{d^dp}{(2\pi)^d}$$

(note! this converges absolutely !) and writing $\int_{\mathbf{R}}dp_\mu f(p_\mu)=\sum_{n\in\mathbf{Z}}\int_{[-\pi,\pi]}dp_\mu f(p_\mu+2\pi n)$, we have

$$\begin{aligned}C^*(x-y)&=\int_{[-\pi,\pi]^d}e^{ip(x-y)}\hat{C}^*(p)\frac{d^dp}{(2\pi)^d}\\ \hat{C}^*(p)&=\sum_{n\in\mathbf{Z}^d}\frac{1}{(p+2\pi n)^2}\prod_{\mu=1}^d\left(\frac{2\sin p_\mu/2}{p_\mu+2\pi n_\mu}\right)^2.\end{aligned}$$

This is the Gaussian fixed point of the block spin RG. We should now like to perform the linear analysis near C^*. We will do this in a slightly different model, where it is less messy (everything can be done in the case of block spin with identical conclusions).

We will consider, instead of a spin model on $\mathbf{Z}^d$ (lattice cutoff), a “spin model"on $\mathbf{R}^d$ where we use a momentum space cutoff. A priori this is as good from the physics point of view.

Thus, let $d\mu_G$ be the Gaussian measure on $S'(\mathbf{R}^d)$ with covariance

$$G(x-y)=\int e^{ip(x-y)}\frac{\chi(p)}{p^2}\frac{d^dp}{(2\pi)^d}$$

where $d>2$, $\chi(p)$ is a cutoff, say

$$\chi(p)=e^{-p^2}$$

but anything with $\chi(0)=1$ and fast decay at ∞ will do (say $\chi\in S(\mathbf{R}^d)$). A priori we know that only smeared fields $\varphi(f)$, $f\in S(\mathbf{R}^d)$ are integrable w.r.t. μ_G and

$$\int e^{i\varphi(f)}d\mu_G=e^{-\frac{1}{2}(f,Gf)}$$

so that e.g. $\langle\varphi(f)\varphi(g)\rangle=(f,Gg)=\int f(x)g(y)G(x-y)dxdy$. However, now $\varphi(x)$ itself makes sense (defined as $\lim_{\epsilon\to 0}\varphi(f_\epsilon)$ where $\lim_{\epsilon\to0}f_\epsilon=\delta_x$ (e.g. $f_\epsilon(y)=(2\pi\epsilon)^{-d/2}e^{-\frac{1}{2\epsilon}(x-y)^2}$) and we have $\int\varphi(x)\varphi(y)d\mu_G=G(x-y)$ well defined, $\forall x,y$. (Technically, μ_G is supported on *distribution that are C^∞ functions*, see Homework).

Thus, let us consider the finite volume measures

$$d\nu_\Lambda(\varphi)=\frac{1}{Z_\Lambda}e^{-V_\Lambda(\varphi)}d\mu_G(\varphi)$$

where Z_Λ normalizes $\int d\nu_\Lambda = 1, V_\Lambda$ is a function of φ's localized in Λ, e.g., $V_\Lambda = \int_\Lambda dx P(\varphi(x))$ P e.g. a polynomial.
We now describe the RG in this setup. We need to split φ into "local"and "global"or "high momentum"and "low momentum"parts as in the block spin case. We do this by first splitting G : write, for $L > 1$,

$$\hat{G}(p) = \frac{1}{p^2}\chi(p) = \frac{1}{p^2}\chi(Lp) + \frac{1}{p^2}(\chi(p) - \chi(Lp))$$

Or

$$\begin{aligned} G(x-y) &= \int e^{ip(x-y)}\frac{1}{p^2}\chi(Lp)\frac{d^dp}{(2\pi)^d} + \int e^{ip(x-y)}\frac{1}{p^2}(\chi(p)-\chi(Lp)) \\ &= L^{2-d}\int e^{ip\frac{x-y}{L}}\frac{1}{p^2}\chi(p)\frac{d^dp}{(2\pi)^d} + \int e^{ip(x-y)}\frac{1}{p^2}(\chi(p)-\chi(Lp)) \\ &= L^{2-d}G(\frac{x-y}{L}) + \Gamma(x-y) \end{aligned} \tag{14.1}$$

Here $\Gamma(x-y)$ is the "high momentum"or "fluctuation within blocks"part : $\chi(p)-\chi(Lp) = e^{-p^2} - e^{-L^2p^2} = \mathcal{O}(p^2)$ for $p \approx 0$ and it gets its main contribution if $|p| \in [\frac{1}{L}, 1]$. In particular, (prove!) $|\Gamma(x-y)| \leq Ae^{-|x-y|/L}$, i.e. it has correlation length $< \infty$.

Lemma Suppose $\mu_{G_i}, i = 1, 2$ are two Gaussian measures on $S'(\mathbf{R}^d)$ and $G = G_1 + G_2$. Then for all $F \in L^\infty(\mu_G)$,

$$\int F(\varphi)d\mu_G(\varphi) = \int F(\varphi_1 + \varphi_2)d\mu_{G_1}(\varphi_1)d\mu_{G_2}(\varphi_2)$$

Remark We say that φ is the sum of independent Gaussians.
Proof 1°. Holds for $F(\varphi) = e^{i\varphi(f)}$, $f \in S(\mathbf{R}^d) : e^{-\frac{1}{2}(f,Gf)} = \int e^{i\varphi(f)}d\mu_G$ and

$$\int e^{i(\varphi_1(f)+\varphi_2(f))}d\mu_{G_1}d\mu_{G_2} = \int e^{i\varphi_1(f)}d\mu_{G_1}\int e^{i\varphi_2(f)}d\mu_{G_2} = e^{-\frac{1}{2}[(f,G_1f)+(f,G_2f)]} = e^{-\frac{1}{2}(f,Gf)}.$$

2°. Suffices to check for F a cylinder-function i.e. recall cylinder sets

$$C(f_1, \cdots, f_n, A_1, \cdots, A_n) = \{\varphi \in S'(\mathbf{R}^d) | \varphi(f_i) \in A_i, i = 1, \cdots, n\}$$

where $f_i \in S(\mathbf{R}^d)$, A_i Borel sets in $\mathbf{R}$. Let χ_C be the characteristic function of this C. Then our claim follows (why ?) if we prove it for $F = \chi_C$. But $\varphi(f_1), \cdots, \varphi(f_n)$ are Gaussian with covariance

$$\langle\varphi(f_i)\varphi(f_j)\rangle = (f_i, Gf_j) \equiv A_{ij}$$

i.e. for $g : \mathbf{R}^n \to \mathbf{R}$ Borel measurable,

$$\int g(\varphi(f_1), \cdots, \varphi(f_n))d\mu_G(\varphi) = \int g(x_1, \cdots, x_n)d\mu_A(x) \tag{14.2}$$

and similarity for $\varphi_\alpha(f_i)$ $\alpha = 1, 2$:

$$\int g(\varphi_1(f_1) + \varphi_2(f_1), \cdots, \varphi_1(f_n) + \varphi_2(f_n)d\mu_{G_1}d\mu_{G_2}$$

$$= \int g(y_1+z_1,\cdots,y_n+z_n)d\mu_{A_1}(y)d\mu_{A_2}(z) \tag{14.3}$$

But (14.2) = (14.3) if we can show it for $g(x)=e^{i(\lambda,x)}$ for all $\lambda \in \mathbf{R}^n$ i.e. $\int e^{i(\lambda,x)}d\mu_A(x) = \int e^{i(\lambda,y+z)}d\mu_{A_1}(y)d\mu_{A_2}(z)$ which is a special case of 1°

Let us apply the Lemma to (14.1). We note that the covariance of $L^{-\frac{d-2}{2}}\varphi(\frac{x}{L})$ is $L^{2-d}G(\frac{x-y}{L})$ if φ has covariance G. Hence :

$$\int F(\varphi)d\mu_G(\varphi) = \int F(L^{-\frac{d-2}{2}}\varphi(L)+Z)d\mu_G(\varphi)d\mu_\Gamma(Z)$$

This gives our RG :

$$\begin{aligned}
\langle F\rangle_{\nu_\Lambda} &= \int F(\varphi)d\nu_\Lambda(\varphi) = \frac{\int F(\varphi)e^{-V_\Lambda(\varphi)}d\mu_G(\varphi)}{\int e^{-V_\Lambda(\varphi)}d\mu_G(\varphi)} \\
&= \frac{\int F(L^{-\frac{d-2}{2}}\varphi(\frac{\cdot}{L})+Z)e^{-V_\Lambda(L^{-\frac{d-2}{2}}\varphi(L)+Z)}d\mu_G(\varphi)d\mu_\Gamma(Z)}{\int e^{-V_\Lambda(L^{-\frac{d-2}{2}}\varphi(\frac{\cdot}{L})+Z)}d\mu_G d\mu_\Gamma} \\
&= \langle\widetilde{F}\rangle_{\tilde\nu}
\end{aligned}$$

with

$$\begin{aligned}
d\tilde\nu(\varphi) &= \frac{1}{Z}e^{-\widetilde{V}(\varphi)}d\mu_G(\varphi) \qquad Z=\int e^{-\widetilde{V}}d\mu_G \\
e^{-\widetilde{V}(\varphi)} &= \int e^{-V_\Lambda(L^{-\frac{d-2}{2}}\varphi(\frac{\cdot}{L})+Z)}d\mu_\Gamma(Z) \\
\widetilde{F}(\varphi) &= \frac{\int F(L^{-\frac{d-2}{2}}\varphi(\frac{\cdot}{L})+Z)e^{-V_\Lambda(L^{-\frac{d-2}{2}}\varphi(\frac{\cdot}{L})+Z)}d\mu_\Gamma(Z)}{\int e^{-V_\Lambda(L^{-\frac{d-2}{2}}\varphi(\frac{\cdot}{L})+Z)}d\mu_\Gamma(Z)}
\end{aligned}$$

Hence, we got a *renormalized measure* $\tilde\nu$ which is again of same form as ν and we can define the RG as a map of V's by

$$R_L V(\varphi) = \widetilde{V}(\varphi) = -\log\int e^{-V(L^{-\frac{d-2}{2}}\varphi(\frac{\cdot}{L})+Z)}d\mu_\Gamma(Z) \tag{14.4}$$

Compare with Block-spin! The latter can actually, with a suitable definition of φ on the RHS, be written exactly as (14.4).

Remark If $V=\int_\Lambda P(\varphi(x))dx$ then

$$\begin{aligned}
V(L^{-\frac{d-2}{2}}\varphi(L)+Z) &= \int_\Lambda P(L^{-\frac{d-2}{2}}\varphi(\frac{x}{L})+Z(x))dx \\
&= L^d\int_{L^{-1}\Lambda} P(L^{-\frac{d-2}{2}}\varphi(y)+Z(Ly))dy
\end{aligned}$$

i.e. it depends on φ in $L^{-1}\Lambda$. So we may call $\widetilde{V}$ by $\widetilde{V}_{L^{-1}\Lambda}$ if we wish. Of course this will not matter in the limit $\Lambda \nearrow \mathbf{Z}^d$.

We have now a *fixed point* $V=0$, i.e. $d\mu_G \to d\mu_G$ under our RG. Call this the *Gaussian fixed point.*

14.2 Compute $DR_{V=0} = \mathcal{L}$

From (14.4) we get immediately

$$(\mathcal{L}V)(\varphi) = \int V(L^{-\frac{d-2}{2}}\varphi(L) + Z)d\mu_\Gamma(Z)$$

This starts to look not too bad !

Spectrum

Let us first see some examples

Example 1 Let $V = \int_\Lambda \varphi(x)^2 dx$. Then

$$\begin{aligned}
\mathcal{L}V(\varphi) &= \int\int_\Lambda (L^{-\frac{d-2}{2}}\varphi(\frac{x}{L}) + Z(x))^2 dx d\mu_\Gamma(Z) = \\
&= L^{2-d}\int_\Lambda \varphi(\frac{x}{L})^2 dx + \in t_\Lambda dx \int Z(x)^2 d\mu_\Gamma \\
&= L^2 \int_{\Lambda/L} \varphi(x)^2 dx + |\Lambda|\Gamma(0) \quad (\text{since } \int Z(x)^2 d\mu_\Gamma = \Gamma(0)) \\
&= L^2 \int_{\Lambda/L} (\varphi(x)^2 + L^{d-2}\Gamma(0))dx
\end{aligned}$$

Hence, since, from (14.1), $G(0) = L^{2-d}G(0) + \Gamma(0)$, we get

$$\mathcal{L}\int_\Lambda (\varphi(x)^2 - G(0))dx = L^2 \int_{\Lambda/L} (\varphi(x)^2 - G(0))dx$$

i.e. we found an *eigenfunction* (as $\Lambda \to \mathbf{Z}^d$) $\int(\varphi(x)^2 - G(0))dx$ with eigenvalue L^2 i.e. our exponent $\alpha = 2$. Note that this is the mass term

$$\begin{aligned}
&[\lim_{\Lambda\to\mathbf{Z}^d} \frac{1}{Z_\Lambda} e^{-\frac{m^2}{2}\int_\Lambda \varphi(x)^2} d\mu_G(\varphi) = d\mu_{G_{m^2}}(\varphi) \\
&G_{m^2}(x-y) = \int e^{ip(x-y)} e^{-p^2} \frac{1}{p^2 + m^2}]
\end{aligned}$$

This is a relevant eigenvector.

Example 2 Consider $V(\varphi) = \int_\Lambda (\nabla\varphi(x))^2 dx$. In the same way, using $\nabla(\varphi(\frac{x}{L})) = L^{-1}(\nabla\varphi)(\frac{x}{L})$ we get : $\mathcal{L}V = \int_{\Lambda/L}(\nabla\varphi(x))^2 dx + |\Lambda|\nabla^2\Gamma(0)$ and

$$\mathcal{L}\left(\int_\Lambda [(\nabla\varphi(x))^2 - \nabla^2 G(0)]\right) dx = \int_{\Lambda/L} ((\nabla\varphi(x))^2 - \nabla^2 G(0))dx$$

so this is a *marginal* eigenvector : note that it corresponds to a change of the fixed point which is corresponding to $\mathcal{H} = \int(\nabla\varphi)^2$: in fact we have a one-parameter family of fixed points $d\mu_{\alpha G}$ $\alpha > 0$.

Example 3 Consider $V(\varphi) = \int_\Lambda \varphi(x)^4 dx$. Now $(\int d\mu_\Gamma Z^{2m+1} = 0)$

$$\begin{aligned}
&\int d\mu_\Gamma(Z)\left[L^{2(2-d)}\varphi\left(\frac{x}{L}\right)^4 + 6L^{2-d}\varphi\left(\frac{x}{L}\right)^2 Z(x)^2 + Z(x)^4\right] \\
&= L^{2(2-d)}\varphi\left(\frac{x}{L}\right)^4 + 6L^{2-d}\varphi\left(\frac{x}{L}\right)^2 \Gamma(0) + 3\Gamma(0)^2
\end{aligned}$$

so

$$\mathcal{L}V = L^{4-d}\int_{\Lambda/L}\varphi(x)^4dx + 6L^2\Gamma(0)\int_{\Lambda/L}\varphi(x)^2 + 3\Gamma(0)^2|\Lambda|$$

so, putting

$$V_4 = \int_\Lambda(\varphi(x)^4 - 6G(0)\varphi(x)^2 + 3G(0)^2)$$

we get $\mathcal{L}V_4 = L^{4-d}V_4$, i.e. V_4 is

relevant if	$d < 4$
marginal if	$d = 4$
irrelevant if	$d > 4$

It is obvious now that we have eigenvectors V_n, polynomial of degree n, with eigenvalues $L^dL^{n\frac{2-d}{2}}$. A simple way to get the formula is to consider

$$\mathcal{L}e^{\varphi(f)} = e^{L^{\frac{2-d}{2}}\int\varphi(\frac{x}{L})f(x)dx}\int d\mu_\Gamma e^{Z(f)} = e^{\varphi(\tilde{f})+\frac{1}{2}(f,\Gamma f)}$$

where $\tilde{f}(x) = L^{\frac{2-d}{2}}L^df(Lx) = L^{\frac{2+d}{2}}f(Lx)$. Since

$$(\tilde{f},G\tilde{f}) = L^{2+d}\int f(Lx)G(x-y)f(Ly)dxdy = L^{2-d}\int f(x)G(\frac{x-y}{L})f(y)dxdy$$

we have, see (14.1), $(f,\Gamma f) = (f,Gf) - (\tilde{f},G\tilde{f})$ and so

$$\mathcal{L}e^{-\frac{1}{2}(f,Gf)+\varphi(f)} = e^{-\frac{1}{2}(\tilde{f},G\tilde{f})+\varphi(\tilde{f})}$$

This is a kind of generating functional of eigenvectors :
Let

$$H_n(t,a) = \frac{d^n}{d\lambda^n}\,|_{\lambda=0}\; e^{-\frac{1}{2}a\lambda^2+\lambda t}$$

be the n^{th} Hermite polynomial (polynomial of degree n in t, homogenous of degree n in $t, \sqrt{a}$).
Then, let

$$V_n(\varphi,f) = H_n(\varphi(f),(f,Gf)) = \frac{d^n}{d\lambda^n}\,|_{\lambda=0}\; e^{-(f,Gf)\lambda^2+\lambda\varphi(f)}$$

so,

$$\mathcal{L}V_n(\varphi,f) = V_n(\varphi,\tilde{f})$$

Taking $\varphi(f) \to \varphi(x), (f,Gf) \to G(x-x) = G(0)$, $V_n(\varphi,f)$ becomes a polynomial $P_n(\varphi(x)) = \varphi(x)^n + a_1\varphi(x)^{n-2}G(0) + \cdots$ and

$$\mathcal{L}\int_\Lambda P_n(\varphi(x))dx = L^{d+n\frac{2-d}{2}}\int_{\Lambda/L}P_n(\varphi(x))dx$$

where P_n are Hermite polynomials.

A local V is of the form $\int_\Lambda$ (polynomial in $\varphi(x)$ and its derivatives) dx. Clearly we get eigenvectors

$$\int_\Lambda \varphi(x)^n \prod_{k=1}^{\ell} (\nabla^k \varphi(x))^{n_k} dx + \text{lower order}$$

with eigenvalue

$$L^d L^{\frac{2-d}{2}(n+\sum n_k)} L^{-(\sum k n_k)}$$

i.e. the more there are φ's or derivatives, the more V is irrelevant. In the class of even local V's the relevant ones are, for $d \geq 3$:

$\int \varphi^2$	L^2	always
$\int \varphi^4$	L^{4-d}	$d < 4$

the marginal one are :

$\int (\nabla\varphi)^2$	always
$\int \varphi^6$	$d = 3$
$\int \varphi^4$	$d = 4$

In case of odd V, we have $\int \varphi \; L^{\frac{d+2}{2}}$ always relevant (this corresponds to a magnetic field), $\int \varphi^3 \; L^{\frac{6-d}{2}}$ relevant for $d < 6$, $\int \varphi^5 \; L^{5-\frac{3d}{2}}$, relevant $3d < 10$. (note that $\int \nabla\varphi, \int \nabla\varphi\varphi^2$ are boundary terms).

14.3 The space $\mathcal{K}$ for $\mathcal{L}$

We can set up a space $\mathcal{L} : \mathcal{K} \to \mathcal{K}$ and study the spectrum.
Basically, we want to consider V's of the form

$$\int dx_1 \cdots dx_n K(x_1, \cdots, x_n) \prod_{i=1}^{n} \varphi(x_i)$$

and specify the K's. For this, generalize a bit what we found above. Define the *normal ordered* product

$$: \prod_{i=1}^{n} \varphi(x_i) := \sum_{I \subset \{1,\cdots,n\}} (-1)^{\frac{|I|}{2}} \prod_{j \notin I} \varphi(x_j) \langle \prod_{i \in I} \varphi(x_i) \rangle$$

Homework

$$\int \prod_{i=1}^{n} dx_i : \prod_{i=1}^{n} \varphi(x_i) : \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{\partial}{\partial \lambda_i} |_{\lambda_i = 0} \exp[-\frac{1}{2}(f, Gf) + \varphi(f)]$$

$$f = \sum_{i=1}^{n} \lambda_i f_i, \;\; f_i \in S(\mathbf{R}^d)$$

This implies :

$$\mathcal{L} : \prod_{i=1}^{n} \varphi(x_i) := L^{\frac{2-d}{2}n} : \prod_{i=1}^{n} \varphi(\frac{x_i}{L}) :$$

The space $\mathcal{K}_\gamma$ consists of *finite* sums of the form (Let $\varphi \in S(\mathbf{R}^d)$)

$$V(\varphi) = \sum_{\underset{\sim}{n}\ \underset{\sim}{m}} \int K_{\underset{\sim}{n}\ \underset{\sim}{m}}(x_1 \cdots x_N) : \prod_{i=1}^N \varphi(x_i)^{n_i} \nabla\varphi(x_i)^{m_i} : \prod_{i=1}^N dx_i \tag{14.5}$$

where $\underset{\sim}{n} = (n_1, \cdots, n_N)$, $\underset{\sim}{m} = (m_1, \cdots, m_N), n_i \in \mathbf{N}, m_i \in \mathbf{N}^d$, $(\nabla\varphi)^{m_i} = \prod_{\alpha=1}^d (\nabla_\alpha \varphi)^{m_{i\alpha}}$, $m_i = (m_{i1}, \cdots, m_{id})$ and $K_{\underset{\sim}{n}\underset{\sim}{m}}(x_1 \cdots x_N) = K_{\underset{\sim}{n}\underset{\sim}{m}}(x_1 + x, x_2 + x, \cdots, x_N + x)\ \forall x \in \mathbf{R}^d$, (translation invariance) and

$$\int |K_{\underset{\sim}{n}\underset{\sim}{m}}(0, x_2, \cdots, x_N)| e^{\gamma \mathcal{L}(0, x_2, \cdots, x_N)} \equiv \|K_{\underset{\sim}{n}\underset{\sim}{m}}\|_\gamma < \infty$$

where $\mathcal{L}(x_1, \cdots, x_N) \equiv d(\underset{\sim}{x})$ is the length of the shortest connected path Γ in $\mathbf{R}^d$ such that all x_i belong to Γ; here $\gamma > 0$.
Then $\mathcal{L}V$ has kernels K':

$$K'_{\underset{\sim}{n}\underset{\sim}{m}}(\underset{\sim}{x}) = L^{\frac{2-d}{2}\sum(n_i + m_i)} L^{Nd} K_{\underset{\sim}{n}\underset{\sim}{m}}(L\underset{\sim}{x}) \cdot L^{-\sum m_i}$$

and

$$\begin{aligned}
\|K'_{\underset{\sim}{n}\underset{\sim}{m}}\|_{\gamma'} &= L^{\frac{2-d}{2}\sum(n_i+m_i)} L^{-\sum m_i} L^{Nd} \int |K_{\underset{\sim}{n}\underset{\sim}{m}}(L\underset{\sim}{x})| e^{\gamma' d(\underset{\sim}{x})} dx_2 \cdots dx_n \\
&= L^{\frac{2-d}{2}\sum(n_i+m_i)} L^{-\sum m_i} L^{d} \int |K_{\underset{\sim}{n}\underset{\sim}{m}}(\underset{\sim}{x})| e^{\gamma' d(\underset{\sim}{x} L^{-1})} dx_2 \cdots dx_n
\end{aligned}$$

But $d(L^{-1}\underset{\sim}{x}) = L^{-1} d(\underset{\sim}{x})$, so

$$\|K'_{\underset{\sim}{n}\underset{\sim}{m}}\|_{L\gamma} \leq L^d L^{\frac{2-d}{2}\sum_{i=1}(n_i+m_i)} L^{-\sum m_i} \|K_{\underset{\sim}{n}\underset{\sim}{m}}\|_\gamma$$

i.e. $\mathcal{L}V \in \mathcal{K}_{L\gamma}$ i.e. in $\mathcal{K}_\gamma$ too.
Thus only (let V be even)

$$\begin{aligned}
&\int K(x-y)\varphi(x)\varphi(y), \int K_{\alpha\beta}(x-y)\nabla_\alpha\varphi(x)\nabla_\beta\varphi(y), \int K(x-y)\varphi(x)\Delta\varphi(y) \\
&\int K(x_1 \cdots x_4)\varphi(x_1)\cdots\varphi(x_4), \int K(x-y)\varphi(x)^2\varphi(y)^2 \\
&\int K(x_1 x_2 x_3)\varphi(x_1)\varphi(x_2)\varphi(x_3)^2, \int K(x-y)\varphi(x)\varphi(y)^3
\end{aligned} \tag{14.6}$$

are not irrelevant in addition to $\int \varphi^2, \int \varphi^4, \int (\nabla\varphi)^2$. These can be split into relevant/marginal and irrelevant as follows. Consider e.g. the first term in (14.6). Let $\widehat{K}(p)$ be the Fourier transform of K. Since $\int |K(x)| e^{\gamma|x|} dx < \infty$ we get $\widehat{K}(p)$ is analytic in $|\mathrm{Im} p| < \gamma$. For simplicity we consider only rotation invariant V's i.e. $K(x) = K(|x|)$.

Lemma Let $K(x)$ be rotation invariant, $\int |K(x)| e^{\gamma|x|} dx < \infty$. Then

$$\int K(x-y)\varphi(x)\varphi(y) dx dy = \alpha \int \varphi(x)^2 + \beta(\nabla\varphi(x))^2 + \tilde{V}(\varphi)$$

where $\tilde{V} \in K_{\gamma/2}$ is irrelevant ($\varphi \in S(\mathbf{R}^d)$).

<u>Proof</u> Since $\int |K(x)| e^{\gamma|x|} dx < \infty$ we get

$$|\widehat{K}(p)| = |\int e^{ipx} K(x) dx| \leq \int e^{|\mathrm{Im} p||x|} |K(x)| dx$$

is analytic in $|\mathrm{Im} p| < \gamma$. Taylor expand

$$\widehat{K}(p) = \widehat{K}(0) + \beta p^2 + \hat{R}(p) \tag{14.7}$$

(we used the rotation symmetry to get the second term of the form βp^2) where

$$\begin{aligned}\hat{R}(p) &= \frac{1}{6}\int_0^1 dt(1-t)^3 \frac{d^4}{dt^4}\widehat{K}(tp)\\ &= \sum_{\alpha\beta\gamma\delta} p_\alpha p_\beta p_\gamma p_\delta \widehat{H}_{\alpha\beta\gamma\delta}(p)\end{aligned}$$

with $|\widehat{H}_{\alpha\beta\gamma\delta}(p)| \leq C$ for $|\mathrm{Im} p| < \gamma$. Thus,

$$H_{\alpha\beta\gamma\delta}(x) \equiv \int e^{-ipx}\widehat{H}_{\alpha\beta\gamma\delta}(p) d^d p/(2\pi)^d$$

satisfies $\int e^{\frac{\gamma}{2}|x|}|H_{\alpha\beta\gamma\delta}(x)|dx < \infty$. Since under Fourier transform $p_\alpha \to +i\frac{\partial}{\partial x_\alpha}$

$$\int K(x-y)\varphi(x)\varphi(y) = \alpha\int\varphi(x)^2 + \beta\int(\nabla\varphi)^2 + \tilde{V}$$

where $\alpha = \widehat{K}(0), \beta = (\beta \text{ in } (14.7))$ and $\tilde{V}$ has 4 derivatives in φ's i.e. $L^d L^{\frac{2-d}{2}\sum(n_i+m_i)-\sum m_i}$ $= L^{-d}$.

The other terms can be analyzed similarly, e.g.

$$\int K(x_1,\cdots,x_4)\prod\varphi(x_i) = a\int\varphi^4 + \tilde{V}$$

where $a = \int K(0,x_2,x_3,x_4)dx_2dx_3dx_4$ and $\tilde{V}$ is irrelevant. Thus every $V \in \mathcal{K}_\gamma$ can be written as

$$V = r\int:\varphi^2: + z\int:(\nabla\varphi)^2: + \lambda\int:\varphi^4: + \tilde{V} \quad , \quad \tilde{V} \in \mathcal{K}_{\gamma/2}$$

and then

$$\mathcal{L}V = L^2 r\int:\varphi^2: + z\int:(\nabla\varphi)^2: + L^{4-d}\lambda\int:\varphi^4: + \tilde{V}'$$

with $\tilde{V}' \in \mathcal{K}_{L\gamma/2}$, $\|\tilde{V}'\|_{L\gamma/2} \leq L^{-2}\|\tilde{V}\|$. These are our “coordinates"in $\mathcal{K}$.

15 Perturbative analysis of RV

15.1 General formalism

Consider now the full R:

$$RV(\varphi) = -\log\int d\mu_\Gamma(Z)\exp[-V(L^{\frac{2-d}{2}}\varphi(\frac{\cdot}{L}) + Z)] \tag{15.1}$$

Suppose V is in $\mathcal{K}$ as above. There are several problems :

1. φ in (15.1) lives on $\mathbf{R}^d$ and is not in $S(\mathbf{R}^d)$, rather it is in $S' \cap C^\infty$ i.e. it may increase as $x \to \infty$. This is the **IR**-problem. We used Λ for that, so we may replace the kernels

$K_{\underset{\sim}{n}\ \underset{\sim}{m}}$ above by $K_{\underset{\sim}{n}\ \underset{\sim}{m}}\ \chi$ where χ restricts $x_i \in \Lambda$.

2. More serious problem is the convergence of (15.1) as Z becomes large. This is the *stability problem.* Say for $V = \lambda\varphi^4$ we need at least Re $\lambda > 0$. For a V as in previous section this positivity property is not easy to state. Instead of trying to address the stability problem, we will calculate RV perturbatively in λ if $V = \lambda\varphi^4$. With a lot of work, RV can actually be analyzed rigorously. One does the following :

a) For $|\varphi(x)| < C$ one uses the representation in terms of $K_{\underset{\sim}{n}\ \underset{\sim}{m}}$, actually an infinite series.

b) For $|\varphi(x)| > C$ one uses another representation where positivity is explicit.

The problem is to combine a) et b) : $\varphi(x)$ can be small and $\varphi(y)$ large nearby. One uses expansion ideas from statistical mechanics to decouple these problems.

Thus, let us start with $V(\varphi) = \lambda \int_\Lambda \varphi(x)^4 dx$ and see what RV looks like. Expand in powers of λ:

$(RV)(\varphi) = \sum_{n=0}^{N} \lambda^n (RV)_n(\varphi) +$ Remainder

$(RV)_n(\varphi) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\ |_{\lambda=0}\ (RV)(\varphi) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\ |_{\lambda=0}\ \ (-\log \int e^{-V(L^{\frac{2-d}{2}}\varphi(L+Z)}d\mu_\Gamma(Z)).$

Hence,

$$
\begin{aligned}
\lambda(RV)_1(\varphi) &= \lambda \int\int_\Lambda (L^{\frac{2-d}{2}}(\varphi(\frac{x}{L}) + Z(x))^4 dx d\mu_\Gamma = \mathcal{L}V(\varphi) \\
\lambda^2(RV)_2(\varphi) &= -\frac{1}{2}[\int V^2 d\mu_\Gamma - \int V d\mu_\Gamma \int V d\mu_\Gamma] \\
&= -\frac{1}{2}\langle V(L^{\frac{2-d}{2}}\varphi + Z); V(L^{\frac{2-d}{2}}\varphi + Z)\rangle_\Gamma
\end{aligned}
$$

where we use the notation of truncated expectations.

In general

$$\lambda^n(\mathrm{RV})_n(\varphi) = (-)^{n-1}\frac{1}{n!}\langle V; V; \cdots; V\rangle_\Gamma$$

Consider $(\mathrm{RV})_2$: Denote $L^{\frac{2-d}{2}}\varphi(\frac{x}{L}) \equiv \Psi(x)$. So

$$
\begin{aligned}
&\langle(\Psi(x)+Z(x))^4(\Psi(y)+Z(y))^4\rangle_\mu - \langle(\Psi(x)+Z(x))^4\rangle_\mu\langle(\Psi(y)+Z(y))^4\rangle_\mu \qquad (15.2)\\
&= 36.2.\Psi(x)^2\Psi(y)^2\Gamma(x-y)^2 + 16.3(\Psi(x)\Psi(y)^3 + \Psi(x)^3\Psi(y))\Gamma(x-y)\Gamma(0) \\
&+4!\Gamma(x-y)^4 + \begin{pmatrix} 4 \\ 2 \end{pmatrix}^2 \cdot 2\Gamma(x-y)^2\Gamma(0)^2 \\
&+16.3!\Psi(x)\Psi(y)\Gamma(x-y)^3 + 6.4.3(\Psi(x)^2+\Psi(y)^2)\Gamma(x-y)\Gamma(0) \\
&+16\Psi(x)^3\Psi(y)^3\Gamma(x-y)
\end{aligned}
$$

or graphically

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where we have same rules as before <u>except that</u> :

- lines are $\Gamma(x-y)$

- external legs have $\Psi(x) = L^{\frac{2-d}{2}}\varphi(\frac{x}{L})$.

Thus we get : $(\mathrm{RV})_n(\varphi)$ = sum of all *connected* graphs with n vertices, and m external legs $0 \leq m \leq n$ and $L^{\frac{2-d}{2}}\varphi(\frac{x}{L})$ attached at each leg, $\Gamma(x-y)$ at each line. Consider such a connected graph. Recall that $|\Gamma(x-y)| \leq Ce^{-\alpha|x-y|}$, for some α. Thus our graphs give terms

$$(RV)_n(\varphi) = \sum \int K_{\underset{\sim}{n}\ \underset{\sim}{m}}(\underset{\sim}{x}) \prod \varphi(x_i)$$

where the kernels satisfy $\|K_{\underset{\sim}{n}\ \underset{\sim}{m}}\|_\gamma < \infty$, for some γ. Thus $\sum_{n=0}^N \lambda^n (RV)_n(\varphi) \in \mathcal{K}_\gamma$ for some γ.

It is more convenient to consider $V(\varphi) = \lambda \int_\Lambda : \varphi(x)^4 : dx$ where, recall, $: \varphi^4(x) := \varphi(x)^4 - 6\varphi(x)^2\Gamma(0) + 3\Gamma(0)^2$. Then

$$\lambda\ (\mathrm{RV})_1(\varphi) = \mathcal{L}V = \lambda L^{4-d} \int_{\Lambda/L} : \varphi(x)^4 :$$

and $(RV)_n$ *has graphs with no tadpole-pieces.*

<u>Homework : Prove this !</u>

Thus, e.g. RV_2 = PICTURE

From our analysis in previous section, we get (for $d = 4$)

$$(RV)(\varphi) = (\lambda + a\lambda^2 + \mathcal{O}(\lambda^3)) \int_{\Lambda/L} : \varphi(x)^4 : + (b\lambda^2 + \mathcal{O}(\lambda^3)) \int_{\Lambda/L} : \varphi^2 :$$
$$+ (c\lambda^2 + \mathcal{O}(\lambda^3)) \int_\Lambda (\nabla\varphi)^2 + \widetilde{V}$$

where $\widetilde{V}$ is *irrelevant* i.e. $\mathcal{L}\widetilde{V}$ contracts and $\widetilde{V} = \mathcal{O}(\lambda^2)$. Here a comes from PICTURE and PICTURE, $a = -\frac{72}{2}\int \Gamma(x)^2 dx + \frac{16.9}{2}\int \Gamma(x-y)L^{2-d}C(\frac{x}{L} - \frac{y}{L})$ (see computation for $d = 4$ below); b, c from PICTURE and PICTURE and PICTURE (see computation for $d = 4$ below).

<u>Iteration</u>

Suppose now $V = \lambda \int : \varphi^4 : + r \int : \varphi^2 : + z \int : (\nabla\varphi)^2 : + \widetilde{V}$ where $\widetilde{V} \in K$ is irrelevant, $\|\mathcal{L}\widetilde{V}\|_\gamma \leq L^{-\alpha}\|\widetilde{V}\|_\gamma$ for $\alpha > 0$. Calculate perturvatively RV:

$$(RV)(\varphi) = \sum_{n=1}^N (-)^{n+1} \langle V; V; \cdots; V\rangle_\Gamma + \text{Remainder} \tag{15.3}$$

where $V = V(L^{-\frac{d-2}{2}}\varphi(\frac{\cdot}{L}) + Z)$

$$\equiv \sum_{n=0}^N\ (\mathrm{RV})_n(\varphi) + \text{Remainder}$$

The point is now that $RV_n \in K_\gamma$ again. Indeed, we get graphs as before and also from r, z and $\widetilde{V}$. Pictorially, the $\widetilde{V}$ ones are

PICTURE

where PICTURE are $K_{\underset{\sim}{n}\ \underset{\sim}{m}}$ and lines are $\Gamma(x-y)$.

It is not hard to show that the exponential decay of $K_{\underset{\sim}{n}\ \underset{\sim}{m}}$ and Γ give rise to exponential decay of these graphs. Thus the claim.

<u>Example</u> E.g. $\langle (\int K(x_1,\cdots,x_4)\prod_{i=1}^4 \varphi(x_i))^2\rangle$ gives for instance

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$$
\begin{aligned}
&= \int \prod_{i=1}^4 dx_i L^{\frac{2-d}{2}} \varphi(\frac{x_i}{L}) \int \prod_{i=1}^4 dy_i K(x_1x_2y_1y_2)\Gamma(y_1-y_3)\Gamma(y_2-y_4)K(y_3y_4x_3x_4) \\
&\equiv \int \widetilde{K}(x_1\cdots x_4)\prod \varphi(x_i)dx_i
\end{aligned}
$$

where

$$
\widetilde{K}(\underset{\sim}{x}) = L^{4-2d}L^{4d}\int \prod_{i=1}^4 dy_i K(Lx_1Lx_2,y_1y_2)\Gamma(y_1-y_3)\Gamma(y_2-y_4)K(y_3y_4Lx_3Lx_4)
$$

and

$$
\begin{aligned}
&\int e^{\gamma\mathcal{L}(0,x_2,x_3,x_4)}|\widetilde{K}(0,\cdots,x_4)|dx_2\cdots dx_4 \\
&= L^{-3d}\int e^{L^{-1}\gamma\mathcal{L}(0,x_2,x_3,x_4)}|\bar{K}(0,L^{-1}x_2,L^{-1}x_3,L^{-1}x_4)|dx_2\cdots dx_4 \qquad (15.4)
\end{aligned}
$$

Use $\mathcal{L}(0,x_2,x_3,x_4)\le \mathcal{L}(0,x_2,y_1,y_2)+|y_1-y_3|+|y_2-y_4|+\mathcal{L}(y_3,y_4,x_3,x_4)$ (since the RHS is the length of a particular path Γ such that $0,x_2,x_3,x_4\in\Gamma$).

So,

$$
(15.4) \le L^{4-d}\int \prod_{i=1}^4 dy_i \prod_{i=2}^4 dx_i \bar{K}(0,x_2,y_1y_2)\bar{\Gamma}(y_1-y_3)\bar{\Gamma}(y_2-y_4)\bar{K}(y_3,\cdots,x_4)
$$

with $\bar{K}=e^{L^{-1}\gamma\mathcal{L}}K$, $\bar{\Gamma}(x)=e^{L^{-1}\gamma|x|}\Gamma(x)$.

Use $|\bar{\Gamma}(y_2-y_4)|\le C$, integrate over y_4,x_3,x_4 :

$$
(15.4) \le L^{4-d}\int \prod_{i=1}^3 dy_i dx_2|\bar{K}(0,x_2,y_1,y_2)\bar{\Gamma}(y_1-y_3)|\|K\|_{L^{-1}\gamma}
$$

Integrate y_3 : $\int|\bar{\Gamma}(y_1-y_3)|dy_3<C'$ and then the rest :

$$
(15.4)\le CL^{4-d}\|K\|^2_{L^{-1}\gamma}\le CL^{4-d}\|K\|^2_{\gamma}
$$

<u>Summary</u> Perturbatively, the R^nV retains the form:

$$
V_n\equiv R^nV(\varphi)=z_n\int_{L^{-n}\Lambda}:(\nabla\varphi)^2:+r_n\int_{L^{-n}\Lambda}:\varphi^2:+\lambda_n\int_{L^{-n}\Lambda}:\varphi^4:+\widetilde{V}_n
$$

where $\widetilde{V}_n \in \mathcal{K}$ and $\|\mathcal{L}\widetilde{V}_n\| \leq L^{-\alpha}\|V_n\|, \quad \alpha > 0$.
We get the recursion (denote $g_n = (z_n, r_n, \widetilde{V}_n)$)

$$\begin{aligned}
z_{n+1} &= z_n + \mathcal{O}(V_n^2) \\
r_{n+1} &= L^2 r_n + \mathcal{O}(V_n^2) \\
\lambda_{n+1} &= L^{4-d}\lambda_n - a\lambda_n^2 + \mathcal{O}(\lambda_n^3, \lambda_n g_n, g_n^2) \\
\widetilde{V}_{n+1} &= \mathcal{L}\widetilde{V}_n + \mathcal{O}(V_n^2)
\end{aligned}$$

Let us see how these behave as $n \to \infty$.

15.2 $\quad d > 4$

We want to determine the critical point, i.e. take

$$V = r\int :\varphi^2: + \lambda\int :\varphi^4:$$

and find $r(\lambda)$ such that $V \in \mathcal{M}_s$. Since $4 - d < 0$ we expect the relevant fixed point to be the Gaussian $V = 0$.
Hence, the problem is to find $r_0 = r(\lambda)$ such that

$$r_n, \lambda_n, \widetilde{V}_n \to 0 \quad \text{as} \quad n \to \infty$$

Since $r_{n+1} = L^2 r_n + \cdots$ tends to increase we need to be careful. Let us construct r_0 inductively. Given λ (small), take $r_0 \in [-A\lambda^2, A\lambda^2] \equiv I_0, A$ chosen below. Then $|r_1 - L^2 r_0| \leq C\lambda^2 + C(A)\lambda^3$ where C is A-independent. Consider r_1 as a function of r_0. For A large enough, r_1 maps our interval

$$r_1(I_0) \supset \left[-\frac{L^2}{2}A\lambda^2, \frac{L^2}{2}A\lambda^2\right]$$

i.e., since $\lambda_1 \leq L^{4-d}\lambda + \mathcal{O}(\lambda^2) < \lambda$, we can, by continuity, find an interval

$$I_1 \subset I_0$$

such that

$$r_1(I_1) = [-A\lambda_1^2, A\lambda_1^2]$$

Now, keep on iterating. We find intervals $I_n \subset I_{n-1} \subset \cdots I_0$ such that r_n as a function of r_0 satisfies

$$r_n(I_n) = [-A\lambda_n^2, A\lambda_n^2]$$

and $\lambda_n \leq L^{\frac{1}{2}(4-d)n}\lambda$, $\|\widetilde{V}_n\| \leq \mathcal{O}(\lambda_n^2)$.

Thus, $\bigcap_{n=0}^\infty I_n \neq \emptyset$ (since I_n closed $\subset I_{n-1}$) and, for $r_0 \in \bigcap_{n=0}^\infty I_n$, we have

$$\widetilde{V}_n, \lambda_n, r_n \to 0 \quad \text{as} \quad n \to \infty$$

Actually it is readily seen that $|I_n| \xrightarrow[n\to\infty]{} 0$ so $\bigcap_{n=0}^{\infty} I_n$ = one point $\equiv r(\lambda)$, the critical r value. Clearly $r(\lambda) = \mathcal{O}(\lambda^2)$.
How about z_n? Since $z_{n+1} = z_n + \mathcal{O}(\lambda_n^2)$ and $z_0 = 0$ we have $z_n = \sum_{m=0}^{n-1} \mathcal{O}(\lambda_m^2) = \sum_{m=0}^{n-1} \mathcal{O}((L^{(4-d)m}\lambda)^2) \xrightarrow[n\to\infty]{} z_\infty = \mathcal{O}(\lambda^2)$. So

$$R^n V \xrightarrow[n\to\infty]{} z_\infty \int :(\nabla\varphi)^2:$$

Hence, we have the one-parameter family of Gaussian fixed points $e^{-z\int :(\nabla\varphi)^2:} d\mu_G$ and $e^{-r(\lambda)\int :\varphi^2: -\lambda\int :\varphi^4:} d\mu_G \xrightarrow[R^n]{} e^{-z_\infty \int :(\nabla\varphi)^2:} d\mu_G$.
The Gaussian line has one unstable direction $:\varphi^2:$ and once that is fixed to the critical value we flow to the line:

PICTURE

15.3 $d = 4$

Since $\lambda_{n+1} = \lambda_n$ to leading order we need to go to $\mathcal{O}(\lambda^2)$. The sign of a in $\lambda_{n+1} = \lambda_n - a\lambda_n^2 + \cdots$ is very important so let us calculate it.
We will prove that $\lambda_n \to 0$ and $r_n, z_n, \widetilde{V}_n = \mathcal{O}(\lambda_n^2)$. Thus a gets contribution from the term

$$-\frac{1}{2}\int \langle: (L^{\frac{2-d}{2}}\varphi(\frac{x}{L}) + Z(x))^4 :: (L^{\frac{2-d}{2}}\varphi(\frac{y}{L}) + Z(y))^4 :\rangle dxdy$$

This gives graph a) PICTURE which contributes directly and b) PICTURE which contributes indirectly.

a) Equals

$$\begin{aligned} &-\frac{72}{2}L^{4-2d}\int dxdy\varphi(\frac{x}{L})^2\varphi(\frac{y}{L})^2\Gamma(x-y)^2 dxdy \\ &= -\frac{72}{2}L^{4-2d}\int dx\varphi(\frac{x}{L})^4(\int dy\Gamma(y)^2) + \text{irrelevant (goes into } \widetilde{V}) \end{aligned}$$

b) We have to normal order PICTURE since $\widetilde{V}_{n+1}$ is normal ordered. This graph equals

$$-\frac{1}{2}4^2 L^{-\frac{2-d}{2}-6}\int :\varphi(\frac{x}{L})^3:\Gamma(x-y):\varphi(\frac{y}{L})^3:$$

Use

$$\begin{aligned} &:\varphi(\frac{x}{L})^3::\varphi(\frac{y}{L})^3: =: \varphi(\frac{x}{L})^3\varphi(\frac{y}{L})^3: \\ &+ 9G(\frac{x-y}{L}):\varphi(\frac{x}{L})^2\varphi(\frac{y}{L})^2: + \text{quadratic term} + \text{constant} \end{aligned}$$

and write

$$\int G(\frac{x-y}{L})\Gamma(x-y):\varphi(\frac{x}{L})^2\varphi(\frac{y}{L})^2: dxdy = \int dx:\varphi(\frac{x}{L})^4:\int dyG(\frac{y}{L})\Gamma(y) + \text{irrelevant}$$

Thus a) and b) together give

$$-36L^{4-d}\int dx:\varphi(x)^4:\int(\Gamma(y)^2+2L^{2-d}G(\frac{y}{L})\Gamma(y))=-b(L)\int:\varphi^4:dx$$

(using $L^{4-d}=1$, $\Gamma(y)^2+2L^{2-d}G(\frac{y}{L})\Gamma(y)=G(y)^2-(L^{2-d}G(\frac{y}{L}))^2$), where

$$b(L)=36\int dy[G(y)^2-(L^{-2}G(\frac{y}{L}))^2], \tag{15.5}$$

Therefore our recursion now is

$$\begin{aligned}
z_{n+1} &= z_n+\mathcal{O}(\lambda_n^2)\\
r_{n+1} &= L^2r_n+\mathcal{O}(\lambda_n^2)\\
\lambda_{n+1} &= \lambda_n-b(L)\lambda_n^2+\mathcal{O}(\lambda_n^3)\\
\widetilde{V}_{n+1} &= \mathcal{L}\widetilde{V}_n+\mathcal{O}(\lambda_n^2)\\
\|\mathcal{L}\widetilde{V}_n\| &\leq L^{-2}\|\widetilde{V}_n\|
\end{aligned} \tag{15.6}$$

How does (15.6) behave?
Consider first the differential equation

$$\frac{d\lambda}{dn}=-\beta\lambda^2$$

so

$$\frac{d\lambda}{\lambda^2}=-\beta n,\ \frac{1}{\lambda_n}=\frac{1}{\lambda}+\beta n,\ \lambda_n=\frac{\lambda}{1+n\beta\lambda}\to 0$$

i.e. for large $n, \lambda_n\sim\frac{1}{n\beta}$.

<u>Homework</u> Prove that (15.6) $\Rightarrow$ $\lambda_n<\frac{c}{n}$. Actually there is some interest in calculating $b(L):b(L)=-36\int d^4y\int_0^{\log L}ds\frac{d}{ds}(e^{-2s}G(e^{-s}\vec{y}))^2$ but

$$\frac{d}{ds}(e^{-2s}G(e^{-s}\vec{y}))^2=(-4-\vec{y}\cdot\vec{\nabla}_y)(e^{-2s}G(e^{-s}\vec{y}))^2$$

where we used the fact that $G(\vec{x})$ depends on $|\vec{x}|:G(\vec{x})=\widetilde{G}(r)\ r=|\vec{x}|$ and

$$\frac{d}{ds}\widetilde{G}(e^{-s}r)=-e^{-s}r\widetilde{G}'(e^{-s}r)=-r\frac{d}{dr}\widetilde{G}(e^{-s}r)=-\vec{x}\cdot\vec{\nabla}_xG(e^{-s}\vec{x}).$$

Thus, (with $A=\int_{S^3}d\Omega$)

$$\begin{aligned}
b(L) &= +36\int_0^{\log L}ds\int_0^\infty dr\ r^3(4+r\frac{d}{dr}(e^{-2s}G(e^{-s}r))^2)\int_{S^3}d\Omega\\
&= 36A\int_0^{\log L}ds\int_0^\infty dr\frac{d}{dr}(r^4(e^{-2s}G(e^{-s}r))^2)\\
&= 36A\int_0^{\log L}ds\lim_{r\to\infty}r^4(e^{-2s}G(e^{-s}r))^2
\end{aligned}$$

But

$$G(r) = \int \frac{d^4p}{(2r)^4} \frac{e^{ip_1 r}}{p^2} e^{-p^2} \tag{15.7}$$
$$= r^{-2} \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip_1}}{p^2} e^{-p^2/r^2} \xrightarrow[r\to\infty]{} r^{-2} \int \frac{d^3p}{(2\pi)^3} \frac{e^{-\sqrt{p^2}}}{2\sqrt{p^2}}$$

So

$$\lim_{r\to\infty} r^4 (e^{-2s} G(e^{-s}r))^2 = \left(\int_0^\infty \frac{d\rho}{2\pi^2} \rho e^{-\rho} \right)^2 = \left(\frac{1}{2\pi^2} \right)^2$$

Hence

$$b(L) = 9A\pi^{-4} \log L \equiv \beta_2 \log L$$

A = area of unit sphere in $\mathbf{R}^4$.

Remark β_2 is *universal*, did not depend on our cutoff e^{-p^2}, could have used any cutoff $\chi(p)$! This is a deep fact, see below.

Summary For $d = 4$, again we find a $r_0(\lambda)$ such that

$$R^n(r_0 \int :\varphi^2: + \lambda \int :\varphi^4:) = V_n$$

has $\lambda_n \to 0$ like $\frac{1}{n}$, $r_n = \mathcal{O}(\lambda_n^2) = \mathcal{O}(\frac{1}{n^2}) \to 0$ $\tilde{V}_n = \mathcal{O}(\lambda_n^2) \to 0$ and

$$z_n \to z_\infty = \sum_{n=0}^{\infty} \mathcal{O}(\lambda_n^2) < \infty$$

since $\sum \frac{1}{n^2}$ converges. We go to the Gaussian fixed point, but only *logarithmically* in the scale: $n \sim \log L^n$.

15.4 $d < 4$

Now $\lambda_n = L^{4-d}\lambda_n + \mathcal{O}(\lambda_n^2)$ *increases*, so sooner or later λ_n is big enough so we can not trust our perturbative analysis that holds only if λ_n is small enough.
It is very instructive to pretend that d is a continuous variable and take $d = 4 - \epsilon$, ϵ small. Then

$$\lambda_{n+1} = L^\epsilon \lambda_n - b(L)\lambda_n^2 + \mathcal{O}(\lambda_n^3)$$

Let us look for a fixed point $\lambda^* = a\epsilon + \mathcal{O}(\epsilon^2)$
Get $1 = L^\epsilon - b(L)a\epsilon + \mathcal{O}(\epsilon^2)$
or $a = \frac{\log L}{b(L)} = \beta_2$ above (we used $b(L)\mid_{d=4-\epsilon} = b(L)\mid_{d=4} + \mathcal{O}(\epsilon) = \beta_2 \log L + \mathcal{O}(\epsilon)$). Thus a fixed point would be $\lambda^* = \beta_2\epsilon + \mathcal{O}(\epsilon^2)$.
BUT now $z_{n+1} = z_n + \mathcal{O}(\lambda_n^2) = z_n + \mathcal{O}(\lambda^{*2})$ *increases*. What went wrong? We have indeed a new fixed point $\lambda^* \neq 0$ and we need a different scaling of φ in our RG :

$$L^{-\frac{d-2}{2}} \to L^{-\frac{d-2+\eta}{2}}$$

Let us proceed formally. Now

$$\begin{aligned}
& e^{-\mathcal{H}} = e^{-\frac{1}{2}(\varphi, G^{-1}\varphi) - V} \approx e^{-\frac{1}{2}\int(\nabla\varphi)^2 - V} \\
& \to e^{-L^{-\eta}\frac{1}{2}\int(\nabla\varphi)^2 - RV(L^{-\frac{\eta}{2}}\varphi)} \\
& = e^{-\frac{1}{2}(L^{-\eta}+\mathcal{O}(\lambda^2))\int(\nabla\varphi)^2 - r_1\int:\varphi^2: - \lambda_1\int:\varphi^4: - \widetilde{V}}
\end{aligned}$$

Thus, $1 + z_0 = 1$ $(z_0 = 0)$

$$1 + z_{n+1} = L^{-\eta}(1 + z_n) + \mathcal{O}(\lambda_n^2)$$

$\Rightarrow \eta = \mathcal{O}(\epsilon^2)$
and

$$\begin{aligned}
\lambda_{n+1} &= L^{\epsilon-2\eta}\lambda_n - \beta_2 \log L\lambda_n^2 + \mathcal{O}(\lambda_n^3) \\
&= L^{\epsilon}\lambda_n + \mathcal{O}(\epsilon^2)\lambda_n - \beta_2 \log L\lambda_n^2 + \mathcal{O}(\lambda_n^3)
\end{aligned}$$

so λ^* is still $\lambda^* = \beta_2\epsilon + \mathcal{O}(\epsilon^2)$.
We got also $r_{n+1} = L^{2-\eta}r_n + \mathcal{O}(\lambda_n^2)$, $\|\mathcal{L}\widetilde{V}_n\| \leq L^{-2-\eta}\|\widetilde{V}_n\|$

<u>Summary $d = 4 - \epsilon$</u>
Now we have a new fixed point

$$\mathcal{H}^* = \frac{1}{2}\int(\nabla\varphi)^2 + r^*\int:\varphi^2: + \lambda^*\int:\varphi^4: + \widetilde{V}^*$$

with $\lambda^* = \beta_2\epsilon + \mathcal{O}(\epsilon^2)$, $r^* = \mathcal{O}(\epsilon^2)$, $\widetilde{V}^* = \mathcal{O}(\epsilon^2)$.
Also

$$\langle\varphi(x)\varphi(y)\rangle_{\mathcal{H}^*} \sim \frac{1}{|x-y|^{d-2+\eta}}$$

$\eta = \mathcal{O}(\epsilon^2)$.

<u>Remark</u> $d = 3$ is $\epsilon = 1$ i.e. is non-perturbative. We expect, say for block-spin RG, that for all λ there is $r(\lambda)$ such that $\mathcal{H} = \sum(\varphi(x) - \varphi(y))^2 + r(\lambda)\sum\varphi(x)^2 + \lambda\sum\varphi(x)^4$ is critical and $\exists\eta$ such that

$$R^n\mathcal{H} \to \mathcal{H}^*$$

where R^n has $C\varphi(y) = L^{+\frac{d-2+\eta}{2}}\frac{1}{L^d}\sum\varphi(Lx+y)$ $(d = 3)$ and $\eta, \mathcal{H}^*$ are λ-independent. $\mathcal{H}^*$ has only one relevant direction, namely the correlation length and no marginal ones (for small ϵ, one sees that the marginal $(\nabla\varphi)^2$ becomes irrelevant).

16 The Continuum Limit

16.1 Effective Field Theories

We finally consider the QFT problem from the RG point of view. Suppose we have a scalar QFT i.e. a measure ν on $S'(\mathbf{R}^d)$. Let σ_i^n be L^{-n} cube centered at $L^{-n}i, i \in \mathbf{Z}^d$. Suppose $L^{+nd}\int_{\sigma_i^n}\varphi(x)dx \equiv \varphi_n(L^{-n}i)$ are well defined random variables (for the free field they are, the covariance is just $L^{2dn}\int_{\sigma_i^n}dx\int_{\sigma_j^n}dyG(x-y)$), i.e. given a $f \in \mathbf{s}(\mathbf{Z}^d)$ suppose

$$S_n(f) = \int e^{i\sum f_j\varphi_n(L^{-n}j)L^{-nd}}d\nu(\varphi)$$

is a positive definite function (this means that $S(f) = \int e^{i\varphi(f)}d\nu$ extends from $f \in S(\mathbf{R}^d)$ to $f = \sum f_i\chi_{\sigma_i^n}$, χ characteristic function). Then S_n defines a measure on $\mathbf{s}'(L^{-n}\mathbf{Z}^d)$, call it ν_n which is the joint distribution of the variables $\varphi_n(L^{-n}i)$. We call ν_n the *effective QFT on scale* L^{-n} corresponding to ν. Thus, the correlation functions of the effective theory are just averages of those of ν over cubes σ_i^n :

$$\langle\prod_{\alpha=1}^N\varphi_n(L^{-n}i_\alpha)\rangle_{\nu_n} = L^{Ndn}\prod_{\alpha=1}^N\int_{\sigma_{i_\alpha}^n}dx_{i_\alpha}\langle\prod_{\alpha=1}^N\varphi(x_{i_\alpha})\rangle_\nu \tag{16.1}$$

It is convenient to relate all of these effective theories to a fixed lattice $\mathbf{Z}^d$. For this, define a $\mathbf{Z}^d$ field $\phi_n(i) = L^{-an}\varphi_n(L^{-n}i)$ where a is a parameter to be fixed later, and let μ_n be the distribution of ϕ_n, i.e.

$$\langle\prod_\alpha\phi_n(i_\alpha)\rangle_{\mu_n} = \langle\prod_\alpha L^{-an}\varphi_n(L^{-n}i_\alpha)\rangle_{\nu_n} \tag{16.2}$$

How are the different ϕ_n's related? Easy:

$$\begin{aligned}\phi_{n-1}(i) &= L^{-a(n-1)}\varphi_{n-1}(L^{-n+1}i) = L^{-a(n-1)}L^{(n-1)d}\int_{\sigma_i^{n-1}}\varphi(x)dx \\ &= L^{-a(n-1)}L^{-d}\sum_{|j_\alpha|<L/2}L^{nd}\int_{\sigma_{Li+j}^n}\varphi(x)dx = L^{a-d}\sum_j\phi_n(Li+j) \\ &= (C\phi_n)(i)\end{aligned} \tag{16.3}$$

where C is the block spin operator. Thus

$$\mu_{n-1} = C^*\mu_n$$

Suppose μ_n are given by Hamiltonians $\mathcal{H}_n$. Then

$$\mathcal{H}_{n-m} = R^m\mathcal{H}_n \qquad \forall n, m$$

Definition Let $R\mathcal{H}^* = \mathcal{H}^*$. The *unstable manifold* $\mathcal{M}_u$ of $\mathcal{H}^*$ is

$$\mathcal{M}_u = \{\mathcal{H}|\exists\mathcal{H}_n \underset{n\to\infty}{\longrightarrow}\mathcal{H}^*, R^n\mathcal{H}_n = \mathcal{H}\}$$

Note that $R : \mathcal{M}_u \to \mathcal{M}_u$. Thus, we are led to the conjecture :

There exists $a \in \mathbf{R}$ and $\mathcal{H}^*$ such that $R\mathcal{H}^* = \mathcal{H}^*$ and $\mathcal{H}_n \underset{n\to\infty}{\longrightarrow} \mathcal{H}^*$. Then all $\mathcal{H}_n$ are on the unstable manifold of $\mathcal{H}^*$.

Hence: critical phenomena are on $\mathcal{M}_S$, QFT on $\mathcal{M}_u$!

Example ν = Gaussian measure, covariance $\widehat{G}(p) = \frac{1}{p^2+m^2}$ (free field, mass m). Then ν_n has covariance

$$G_n(L^{-n}i, L^{-n}j) = \int_{\mathbf{R}^d} e^{ipL^{-n}(i-j)}\widehat{G}(p) \mid L^{nd}\int_{\sigma_0^n} e^{ipx}dx \mid^2 \frac{d^dp}{(2\pi)^d}$$

Let $\rho_n(p) = \mid L^{nd}\int_{\sigma_0^n} e^{ipx}dx \mid^2$ and, for $a = \frac{d-2}{2}$, μ_n has covariance C_n:

$$\begin{aligned} C_n(i,j) &= L^{(2-d)n}\int e^{ipL^{-n}(i-j)}\frac{1}{p^2+m^2}\rho_n(p)\frac{d^dp}{(2\pi)^d} \\ &= \int e^{ip(i-j)}\frac{1}{p^2+L^{-2n}m^2}\rho_n(L^np)\frac{d^dp}{(2\pi)^d} \end{aligned}$$

where

$$\rho_n(L^np) = |\int_{\sigma_0^0} e^{ipx}dx|^2 = \prod_{\mu=1}^{d}\left(\frac{2\sin p_{\mu/2}}{p_\mu}\right)^2 \equiv \rho(p)$$

is n-independent. Here C_n are covariances that parametrize the Gaussian part of the unstable manifold of the Gaussian fixed point $C_{n=\infty}$ of the block spin RG. They have correlation length L^nm^{-1}.

Suppose now that we have a RG $R : \mathcal{K} \to \mathcal{K}$ and a fixed point $\mathcal{H}^*$. How would we get a QFT from these data?
Easiest way is the *scaling limit*: For $x_i \neq x_j$ $i \neq j$, consider the limit

$$\begin{aligned} \lim_{n\to\infty} L^{Nan}\langle\prod_{i=1}^{N}\phi(L^nx_i)\rangle_{\mathcal{H}^*} &\equiv \lim_{n\to\infty} G^{(n)}(x_1,\cdots,x_N) \\ &\equiv G(x_1\cdots x_n) \end{aligned} \tag{16.4}$$

We expect this to exist: We expect that for $|y_i - y_i|$ large $\forall i \neq j$, we have for $C(y_1,\cdots,y_N) \equiv \langle\prod\phi(y_i)\rangle_{\mathcal{H}^*}$:

$$\begin{aligned} &|C(y_1+z,y_2,\cdots,y_N) - C(y_1,y_2,\cdots,y_N)| \\ &\approx (\min_{i>1}|y_1-y_i|)^{-\epsilon}|C(y_1\cdots y_N)| \end{aligned}$$

$\epsilon > 0$, if $|z| < \mathcal{O}(1)$ (say L). Then

$$\langle\prod_{i=1}^{N}\phi(L^nx_i)\rangle_{\mathcal{H}^*} = \langle\prod_{i=1}^{N}\phi_{av}(L^nx_i)\rangle_{\mathcal{H}^*}(1+\mathcal{O}(L^{-n\epsilon}))$$

where $\phi_{av}(Lx) = L^{-d}\sum_{|y_\alpha|<\frac{L}{2}}\phi(Lx+y)$. Thus,

$$\begin{aligned} G^{(n)}(x_1,\cdots,x_N) &= L^{Nan}\langle\prod_i \phi_{av}(L(L^{n-1}x_i))\rangle_{\mathcal{H}^*}(1+\mathcal{O}(L^{-n\epsilon})) \\ &= G^{(n-1)}(x_1,\cdots,x_N)(1+\mathcal{O}(L^{-n\epsilon})) \end{aligned}$$

since $(C\phi)(x) = L^a\phi_{av}(Lx)$ and $R\mathcal{H}^* = \mathcal{H}^*$. Thus, one expects the limit in (16.4) to exist. The distributions $G(x_1,\cdots,x_N)$ are correlations of a QFT ($x_i \in \mathbf{R}^d$ now). What are the effective field theories ν_m or μ_m or $\mathcal{H}_m$ of this QFT?

<u>Answer</u> All $\mathcal{H}_m$ are just $\mathcal{H}^*$!

Proof This is just bookkeeping. By definition (16.1), (16.2),

$$\langle\prod_{\alpha=1}^N \phi(i_\alpha)\rangle_{\mu_m} = \int dx_1\cdots dx_N\chi(x_\alpha \in \sigma^m_{i_\alpha})G(x_1,\cdots,x_N)L^{N(d-a)m}$$

where $\sigma^m_{i_\alpha} = L^{-m}$ cube at $L^{-m}i_\alpha$.
Write the integral as a Riemann sum and use the definition (16.4) of G:

$$\int\prod dx_\alpha\chi G = \lim_{n\to\infty}\sum_{\{x_\alpha\in L^{-n}\mathbf{Z}^d\}} L^{-Ndn}\chi(x_\alpha\in\sigma^m_{i_\alpha})L^{N(d-a)m}L^{Nan}\langle\prod_{\alpha=1}^N\phi(L^nx_\alpha)\rangle_{\mathcal{H}^*}$$

Write $x_\alpha = L^{-n}y_\alpha$. Then y_α is summed over $y_\alpha \in \mathbf{Z}^d, y_\alpha \in L^{n-m}$-cube at $L^{n-m}i_\alpha$ (take $n > m$).
Thus, using (16.3),

$$\begin{aligned} \langle\prod_{\alpha=1}^N \phi(i_\alpha)\rangle_{\mu_m} &= \lim_{n\to\infty}\langle\prod_\alpha(C^{n-m}\phi)(i_\alpha)\rangle_{\mathcal{H}^*} \\ &= \langle\prod_{\alpha=1}^N\phi(i_\alpha)\rangle_{\mathcal{H}^*} \end{aligned}$$

The scaling limit is a scale-invariant, massless, field theory :

$$G_N(\lambda x_1,\cdots,\lambda x_N) = \lambda^{-Na}G(x_1\cdots x_N)$$

In particular (assuming translation invariance):

$$G_2(x_1,x_2) = \frac{\text{const}}{|x_1-x_2|^a}$$

16.2 Non-Scale Invariant Theories

How to get a QFT with effective theories $\mathcal{H}_n \neq \mathcal{H}^*$? E.g. we would like to get a massive theory. We would like to pose this constructively : find (lattice) cutoff-L^{-n} theories that converge as $n\to\infty$ to a QFT.

Example Look at free field again. Consider Ginzburg-Landau model on $\mathbf{Z}^d$ with correlation length $\sim L^n$:

$$\mathcal{H}_n = \frac{1}{2}\sum(\phi(x)-\phi(y))^2 + \frac{L^{-2n}m^2}{2}\sum\phi(x)^2$$

i.e.

$$\begin{aligned} e^{-\mathcal{H}_n}D\phi &= d\mu_{C_n}(\phi) \\ C_n &= (-\Delta + L^{-2n}m^2)^{-1} \end{aligned}$$

on $\mathbf{Z}^d$. We know that

$$L^{2-\frac{d-2}{2}}\langle\phi(L^n x)\phi(L^n y)\rangle_{C_n} \underset{n\to\infty}{\longrightarrow} \left(\frac{1}{-\Delta_{\mathbf{R}^d}+m^2}\right)(x,y)$$

>From RG we find

$$\begin{aligned} R^{n-m}\mathcal{H}_n &\underset{n\to\infty}{\longrightarrow} \frac{1}{2}(\phi, G_m^{-1}\phi) = \mathcal{H}_m^* \\ G_m(x,y) &= \int_{\mathbf{R}^d} e^{ip(x-y)}\frac{1}{p^2+L^{-2m}m^2}\rho(p)\frac{d^dp}{(2\pi)^d} \qquad (x,y\in\mathbf{Z}^d) \end{aligned}$$

which are on the unstable manifold : $R\mathcal{H}_m^* = \mathcal{H}_{m-1}^*$. Pictorially,

PICTURE

So, in general, we should look for a 1-parameter family of Hamiltonians $\mathcal{H}_{L^n}$ such that $R^{n-m}\mathcal{H}_{L^n}$ converge as $n\to\infty$ for all m. It is easy to see that $\mathcal{H}_{L^n}$ must tend to $\mathcal{M}_S$ as $n\to\infty$ and $R^{n-m}\mathcal{H}_{L^n}$ to $\mathcal{M}_u$.
As an example we consider the φ^4 QFT in various dimensions.

φ^4_d

We consider a momentum cutoff for simplicity and let

$$d\nu_\epsilon(\varphi) = \frac{1}{Z}e^{-z_\epsilon\int_\Lambda :(\nabla\varphi)^2: -\rho_\epsilon\int_\Lambda :\varphi^2: - g_\epsilon\int_\Lambda :\varphi^4:}d\mu_{G_\epsilon}$$

Let $z_\epsilon\int_\Lambda : (\nabla\varphi)^2 : +\rho_\epsilon\int_\Lambda : \varphi^2 : +g_\epsilon\int_\Lambda : \varphi^4 := W_\epsilon(\varphi)$; here[8] $\widehat{G}_\epsilon(p) = \frac{1}{p^2+m^2}\chi(\epsilon p)$ and let, e.g., $\chi = e^{-p^2}$. We ask : are there $z_\epsilon, \rho_\epsilon, g_\epsilon$ such that ν_ϵ converges to a measure ν on S' as $\epsilon\to 0$. We know $\lambda = 0$ is a solution, and ask if a non-Gaussian ν is possible. The effective measures are defined as above, only easier : Write, for $\ell > \epsilon$

$$\begin{aligned} \widehat{G}_\epsilon(p) = \frac{1}{p^2+m^2}\chi(\epsilon p) &= \frac{1}{p^2+m^2}\chi(\ell p) + \frac{1}{p^2+m^2}(\chi(\epsilon p)-\chi(\ell p)) \\ &\equiv \widehat{G}_\ell(p) + \widehat{\Gamma}_{\epsilon\ell}(p) \end{aligned}$$

[8]We put m^2 into the covariance for simplicity : we are concerned with the **UV**-problem, not the **IR**.

and define

$$d\nu_\ell^{(\epsilon)}(\varphi) = \left(\int e^{-W_\epsilon(\varphi+z)} d\mu_{\Gamma_{\epsilon\ell}}(z)\right) d\mu_{G_\ell}(\varphi)$$

which has cutoff ℓ. To connect to Statistical Mechanics and RG, we need to scale as in (16.2):

$$\phi_\epsilon(x) = \epsilon^{\frac{(d-2)}{2}} \varphi(\epsilon x)$$

(we take $a = \frac{d-2}{2}$ to anticipate later). Then ν_ϵ becomes μ_ϵ:

$$d\mu_\epsilon(\phi) = e^{-W_\epsilon(\epsilon^{\frac{2-d}{2}}\phi(\epsilon))} d\mu_{C_\epsilon}(\phi) \equiv e^{-V_\epsilon(\phi)} d\mu_{C_\epsilon}$$

where $\widehat{C}_\epsilon(p) = \frac{1}{p^2+\epsilon^2 m^2}\chi(p)$, **UV**-cutoff 1, Gaussian. (Note the **IR** cutoff)
Here

$$V_\epsilon(\phi) = z_\epsilon \int :(\nabla\varphi)^2: + \epsilon^2 \rho_\epsilon \int :\varphi^2: + \epsilon^{4-d} g_\epsilon \int :\varphi^4:$$

note the powers of ϵ (so called canonical dimensions of mass, coupling). Then we have

$$V_\ell^{(\epsilon)} \equiv R_{\ell/\epsilon} V_\epsilon$$

where R_L is as before:

$$R_L V = -\log \int e^{-V(L^{\frac{2-d}{2}}\phi(L)+Z)} d\mu_\Gamma(Z)$$

$\widehat{\Gamma}(p) = \frac{1}{p^2}(\chi(p) - \chi(Lp))$. Let us put

$$r_\epsilon = \epsilon^2 \rho_\epsilon, \quad \lambda_\epsilon = \epsilon^{4-d} g_\epsilon$$

As before, we could study the iteration $r_n, z_n, \lambda_n \to r_{n+1}, z_{n+1}, \lambda_{n+1}$ but here we can also do it infinitesimally, i.e. as a differential equation. We put $L^n = e^s$ and get for $R_{e^s} V_\epsilon = (z_\epsilon(s), r_\epsilon(s), \lambda_\epsilon(s), \widetilde{V}_\epsilon(s)) \equiv V_\epsilon(s)$:

$$\begin{aligned}
\frac{dz}{ds} &= \mathcal{O}(V^2) \\
\frac{dr}{ds} &= 2r + \mathcal{O}(V^2) \\
\frac{d\lambda}{ds} &= (4-d)\lambda - \beta_2 \lambda^2 + \mathcal{O}(\lambda^3, \lambda(z+r+\widetilde{V}), (z+r+\widetilde{V})^2) \\
\frac{d\widetilde{V}}{ds} &= \mathcal{M}\widetilde{V} + \mathcal{O}(V^2)
\end{aligned}$$

where $\mathcal{M}\widetilde{V}(\phi) = \frac{d}{ds}\,|_0 \int \widetilde{V}(e^{\frac{2-d}{2}s}\phi(e^{-s}\cdot) + Z) d\mu_\Gamma(Z)$.
We want to choose $r_\epsilon(0), z_\epsilon(0), \lambda_\epsilon(0)$ such that $\lim_{t\to\infty} V_{e^{-t}}(t+s) \equiv V^{\text{eff}}(s)$ exists, $\forall s \in \mathbf{R}$.

$\underline{d=2}$ Take $r_\epsilon = 0, \lambda_\epsilon = \epsilon^2 g, z_\epsilon = 0$.

Put

$$\lambda_\epsilon(s) = \epsilon^2 e^{2s} g_t(s), \; r_\epsilon(s) = \epsilon^2 e^{2s} \rho_t(s), \; \epsilon = e^{-t}.$$

We get

$$\frac{dg}{ds} = \mathcal{O}(e^{-2(t-s)}g^2),\ \frac{d\rho}{ds} = \mathcal{O}(e^{-2(t-s)}g^2)$$

Hence the limits $\lim_{t\to\infty} g_t(t+s)$, $\lim_{t\to\infty} r_t(t+s)$ exist. This is a super renormalizable theory : the "bare"coupling g_ϵ can be taken ϵ-independent and $z_\epsilon, \rho_\epsilon$ also.

$\underline{d=3}$ Now $\lambda_\epsilon(s) = \epsilon e^s g_t(s)$ and we have a "resonance"in the ρ-equation:

$$\begin{aligned}\frac{dg}{ds} &= \mathcal{O}(e^{-(t-s)}g^2)\\ \frac{d\rho}{ds} &= \mathcal{O}(g^2)\end{aligned}$$

More precisely, write $\frac{dr}{ds} = 2r + \gamma\lambda^2 + \mathcal{O}(\lambda^3)$, then,

$$\frac{d\rho}{ds} = \gamma g^2 + \mathcal{O}(e^{-(t-s)}g^3)$$

Since g stays almost constant, we get

$$\rho_t(s) = \rho_t(0) + s\gamma g^2 + \text{bounded}$$

Thus, we need to take $\rho_t(0) = -t\gamma g^2$ i.e.

$$g_\epsilon = g,\ \rho_\epsilon = \gamma g^2 \log \epsilon$$

this is a "mass counterterm"of $\mathcal{O}(g^2)$, coming from the graph PICTURE that diverges in **UV** logarithmically.

$\underline{d=4}$ Now we stay in the (r,λ) picture. λ *decreases* logarithmically :

$$\begin{aligned}\frac{d\lambda}{ds} &= -\beta_2\lambda^2 + \mathcal{O}(\lambda^3)\\ \lambda(s) &\sim \frac{\lambda(0)}{1+\beta_2 s\lambda_\epsilon(0)}\end{aligned}$$

If $\lambda_\epsilon(0)$ stays bounded in ϵ as we have to assume in our perturbative analysis, then $\lambda(s) \underset{s\to\infty}{\longrightarrow} 0$.

Thus the only perturbative theory (= one with coupling constant on all scales small) is the Gaussian. Same holds for

$\underline{d>4}$: $\frac{d\lambda}{ds} = -(d-4)\lambda + \mathcal{O}(\lambda^2)$ i.e. $\lambda(s) \to 0$.

One might hope that taking $\lambda_\epsilon(0)$ large enough things might change. E.g. these might be another fixed point λ^* for the RG, i.e. the flow in λ would be

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Unfortunately, this is unlikely to happen : in $d > 4$ it is rigorously proven that for <u>all</u> $z(\epsilon), r(\epsilon), \lambda(\epsilon)$ the only $\epsilon \to 0$ limits are Gaussian. For $d = 4$ there are strong theoretical and numerical arguments.

<u>$d < 4$: nontrivial fixed points</u>

Consider the $d = 3$ QFT constructed above. This has mass $\sim m^2$, coupling λ. What about $m \to 0$? Above we could have taken $m = 0$ and started with $r_\epsilon, \lambda_\epsilon$ to produce an effective Hamiltonian on scale 1 of the form

$$z\int :\nabla\varphi^2: + r\int :\varphi^2: + \lambda\int :\varphi^4: + \widetilde{V}$$

with various values of r. To solve for the **IR** we would need to keep iterating the RG. Now λ increases and eventually, if we really choose $r = r(\lambda) =$ critical point, we would tend to the nontrivial fixed point $\mathcal{H}^*$. Thus, **UV** is controlled by the Gaussian fixed point, **IR** by the non-Gaussian.

We could construct a scaling limit <u>at $\mathcal{H}^*$</u> : this would be a scale-invariant QFT with non-Gaussian **IR** and **UV**. $\mathcal{H}^*$ has (at least) one relevant direction : the correlation length. We could construct, as above, a *massive* QFT corresponding to the unstable manifold of $\mathcal{H}^*$. We expect the unstable manifold of the Gaussian fixed point $\mathcal{H}_0^*$ to be 2-dimensional (r, λ), and of $\mathcal{H}^*$ to be one dimensional. So the flow is

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Thus, a "typical"QFT is a typical point of the r, λ plane with Gaussian **UV** and finite correlation length. The ones on $\mathcal{M}_s(\mathcal{H}^*)$ have non-Gaussian **IR**, Gaussian **IR** and the ones on $\mathcal{M}_u(\mathcal{H}^*)$ have non-Gaussian **UV**, $\xi < \infty$. (We should really include the $r = \infty$ $(\xi = 0)$ "high-temperature"fixed point).

16.3 Asymptotic Freedom

What would have happened if β_2 above had been *negative* at $d = 4$? Now

$$\frac{d\lambda}{ds} = |\beta_2|\lambda^2 + \mathcal{O}(\lambda^3) \Rightarrow \lambda_\epsilon(0) = \frac{\lambda_\epsilon(s)}{1 + |\beta_2| s \lambda_\epsilon(s)}$$

i.e. if $\epsilon = e^{-t}, s = t$, and we want to fix $\lambda_{e^{-t}}(t) = g$, then the "bare coupling"λ_ϵ is

$$\lambda_\epsilon = \frac{g}{1 + |\beta_2| \log \epsilon^{-1} g} \xrightarrow[\epsilon\to 0]{} 0$$

i.e. to have a non-Gaussian theory, we need to let $\lambda_\epsilon \to 0$ as $\epsilon \to 0$. This is obvious, since λ is now *unstable*. Actually, it is instructive to calculate exactly λ_ϵ : we need the λ-equation to $\mathcal{O}(\lambda^3)$ (i.e. again, we need to consider $z, r, \widetilde{V}$ too, to $\mathcal{O}(\lambda^3)$). So consider

$$\frac{d\lambda}{ds} = \beta(\lambda) = \beta_2\lambda^2 + \beta_3\lambda^3 + \tilde{\beta}(\lambda),\ \tilde{\beta} = \mathcal{O}(\lambda^4)$$

So

$$
\begin{aligned}
&\int_{\lambda_\epsilon}^{g} \frac{d\lambda}{\beta(\lambda)} = \log \epsilon^{-1} \\
&= \int_{\lambda_\epsilon}^{g} d\lambda \frac{1}{\beta_2\lambda^2} \frac{1}{1+\frac{\beta_3}{\beta_2}\lambda + \mathcal{O}(\lambda^2)} = \int_{\lambda_\epsilon}^{g} d\lambda \left(\frac{1}{\beta_2\lambda^2} - \frac{\beta_3}{\beta_2^2\lambda} + \mathcal{O}(1) \right) \\
&= \frac{1}{\beta_2\lambda_\epsilon} - \frac{1}{\beta_2 g} - \frac{\beta_3}{\beta_2^2} \log \frac{g}{\lambda_\epsilon} + \mathcal{O}(g) = \log \epsilon^{-1} \\
&\Rightarrow \lambda_\epsilon = \frac{g}{1 + \beta_2 g \log \epsilon^{-1} - \frac{g\beta_3}{\beta_2} \log\log \epsilon^{-1} + \mathcal{O}(1)}
\end{aligned}
$$

This is how the bare coupling should be chosen to have a continuum limit.

Remark We have seen that β_2 is universal (independent on regularization). It can be shown that β_3 is also. (β_4 is not). They are intrinsic, depend only on the continuum limit (which does not depend on the regularization). An example of asympotically free theory is the Ginzburg-Landau model with λ *negative*. This is not stable, but can be rigorously defined by analytic continuation. Since $\lambda = -g$, $g > 0$ we get $\frac{dg}{ds} = \beta_2 g^2 + \cdots \beta_2 > 0$. More interesting one is the $SU(N)$ gauge theory. There one has two couplings z and λ, both marginal, with

$$
\begin{aligned}
\frac{dz}{ds} &= \gamma\lambda^2 + \cdots \\
\frac{d\lambda}{ds} &= \beta_3\lambda^3 + \cdots
\end{aligned}
$$

so $\lambda(s)^2 \approx \frac{g^2}{1+\beta_3 g^2(t-s)}$ where $\epsilon = e^{-t}$, $\lambda_\epsilon^2(t) = g^2$ and $z_\epsilon(t) - z_\epsilon(0) = \gamma \int_0^t \lambda(s)^2 \sim \frac{\gamma}{\beta_3} \log t$ so $z_\epsilon(0)$ must diverge as $-\frac{\gamma}{\beta_3} \log\log \epsilon^{-1}$.

Remarks

1. Thus, we expect to find QFT's at fixed points and on their unstable manifolds. For Ginzburg-Landau model, $d < 4$ we find several non-Gaussian ones. For $d \geq 4$ the fixed points and unstable manifolds are Gaussian.
2. Asymptotically free theories have Gaussian fixed points with marginally unstable non-Gaussian direction. Here we have a non-Gaussian continuum limit.
Examples: *Non Abelian gauge theories* with not too many fermions (QCD is one).
Non-linear σ-models in $d = 2$ (e.g. $O(N), N > 2$ spin systems we discussed earlier). Certain $d = 2$ fermionic models (Gross-Neveu model).

3. IR behaviour of asymptotically free models is interesting. Now $\lambda(s) \nearrow$ as s increases. E.g. in non-Abelian gauge and σ-models. These theories have no relevant variable like r in Ginzburg-Landau model that would provide a mass (correlation length). Nevertheless, they are massive ($\xi < \infty$) : ξ with be $\sim e^{-\text{const}/\lambda^2}$ (so called dimensional transmutation). There is no other critical point than $\lambda = 0$. (λ^2 is the temperature in Heisenberg model).

4. Theories that are expected to be Gaussian (trivial) are : Ginzburg-Landau model, QED, Standard Model of weak and EM interactions. This sounds paradoxical since QED has a very successful renormalized perturbation theory and is the most accurate known physical theory. The point is that, as long as we do not take ϵ to 0, we can keep $\lambda_\epsilon > 0$ and so $\lambda_\epsilon(s) > 0$. Now if one calculates a correlation function in perturbation theory, the graphs that are convergent integrals of the form $\int_{|p|<\epsilon^{-1}} I(p)dp$ depend on the cutoff like ϵ to some power. But we may take (recall, $\lambda_\epsilon = \frac{\lambda}{1-\beta_2 \log \epsilon^{-1} \lambda}$ has to be small so $((\log \epsilon^{-1})^{-1} \sim \lambda)$

$$\epsilon \sim e^{-\text{const}/\lambda}$$

Hence, the dependence on cutoff is $\sim e^{-\text{const}/\lambda}$, very small for λ small ($\frac{1}{\lambda} = 137$ for QED). We say that all the above theories are good *effective theories* for distances $>> \epsilon$ (in practice, energies $<< 10^{15}$ GeV). For smaller distances one needs to find a new theory (say string theory).